Topology

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June 22, 2024

Contents

1 Topological Spaces and Continuous Maps

1. A *metric space* (X, d) is a set X with a function $d : X^2 \to [0, \infty)$ such that
   - $d(x, y) = 0 \iff x = y$.
   - $d(x, y) = d(y, x)$.
   - $d(x, z) \leq d(x, y) + d(y, z)$.
2. $B(x, \epsilon) = \{x' \in X \mid d(x, x') < \epsilon\}$.
3. $f : (X, d) \to (Y, d)$ is called continuous if $\forall x \in X, \forall \epsilon > 0, \exists \delta > 0, f(B(x, \delta)) \subset B(f(x), \epsilon)$.
4. A *topological space* $(X, \mathscr{G})$ is a set X with a set $\mathscr{G} \subset \mathscr{P}(X)$ of *open sets* such that
   - $\emptyset, X \in \mathscr{G}$.
   - $\{A_\alpha\}_{\alpha \in I} \subset \mathscr{G} \implies \cup_{\alpha \in I} A_\alpha \in \mathscr{G}$.
   - $A_1, A_2 \in \mathscr{G} \implies A_1 \cap A_2 \in \mathscr{G}$.
5. A metric space (X, d) introduces a topology by $U \in \mathscr{G} \iff \forall x \in U, \exists \epsilon > 0, B(x, \epsilon) \subset U$.
6. A map $f : (X, d) \to (Y, d)$ is continuous $\iff \forall A \in \mathscr{G}_Y, f^{-1}(A) \in \mathscr{G}_X$. The right clause is how continuity is defined for maps between topological spaces.
7. In $X = \mathbb{R}^n$, all p-norms $|x|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ define metrics which define topologies; these topologies coincide. This includes the case $p = \infty$ where $|x|_\infty := \max_{i \in [n]} |x_i|$.
8. If $f : (X, \mathscr{G}_X) \to (Y, \mathscr{G}_Y)$ and $g : (Y, \mathscr{G}_X) \to (Z, \mathscr{G}_X)$ are continuous, then $g \circ f : (X, \mathscr{G}_X) \to (Z, \mathscr{G}_Z$ is.
9. A constant map $f : x \mapsto c \in Y$ is continuous.
10. The identity map $\mathrm{id} : (X, \mathscr{G}_1) \to (X, \mathscr{G}_2)$ is continuous if and only if $\mathscr{G}_2 \subset \mathscr{G}_1$
11. We compare two topologies $\mathscr{G}_1, \mathscr{G}_2$ on X as follows:
    - $\mathscr{G}_1$ is *weaker* or *coarser* than $\mathscr{G}_2$ if $\mathscr{G}_1 \subset \mathscr{G}_2$; then $\mathscr{G}_2$ is called *finer* or *stronger* than $\mathscr{G}_1$.
    - These comparisons are qualified with "*strictly*" if the inclusions are strict.

2 Subspaces and Homeomorphisms

1. A set $A \subset X$ is called *closed* if its complement is open.
2. Let $\mathscr{F}$ denote all closed sets of $(X, \mathscr{G})$. Then:
   - $\emptyset, X \in \mathscr{F}$.
   - $\{A_\alpha\}_{\alpha \in I} \subset \mathscr{F} \implies \cap_{\alpha \in I} A_\alpha \in \mathscr{F}$.
   - $A_1, A_2 \in \mathscr{F} \implies A_1 \cup A_2 \in \mathscr{F}$.
3. $f : (X, \mathscr{G}_X) \to (X, \mathscr{G}_Y)$ is continuous $\iff \forall C \in \mathscr{F}_Y, f^{-1}(C) \in \mathscr{F}_X$.
4. If $Y \subset X$ then this induces a *subspace topology* as $\mathscr{G}_Y = \{Y \cap U \mid U \in \mathscr{G}_X\}$.
5. We have $\mathscr{F}_Y = \{Y \cap C \mid C \in \mathscr{F}_X\}$.
6. $i : Y \to X$, $y \mapsto y$ is continuous $(Y, \mathscr{G}_Y) \to (X, \mathscr{G}_X)$.
7. If $Z \subset Y \subset X$, then $\mathscr{G}_Z$ induced by $\mathscr{G}_X$ is the same as $\mathscr{G}_Z$ induced by $\mathscr{G}_Y$ where $\mathscr{G}_Y$ is induced by $\mathscr{G}_X$. In other words,
$$\{Z \cap U \mid U = Y \cap V, V \in \mathscr{G}_X\} = \{Z \cap V \mid V \in \mathscr{G}_X\}$$
8. If $Y \subset X$ is a subset of a metric space, then the subspace topology $(Y, \mathscr{G}_Y)$ coincides with the topology induced by the metric space $(Y, d|_{Y \times Y})$.
9. If $f : X \to Z$ is continuous and $f(X) \subset Y$, then $f : X \to Y$ is continuous.
10. If $f : X \to Z$ is continuous then $f|_Y : Y \to Z$ is continuous.
11. If $\{U_i\}_{i \in I}$ is an open cover of X and $f : X \to Y$ has that $f|_{U_i} : U_i \to Y$ is continuous for each $i \in I$, then f is continuous.
12. If $\{C_i\}_{i=1}^n$ is a finite closed cover of X and $f : X \to Y$ has that $f|_{C_i} : C_i \to Y$ is continuous for each $i \in I$, then f is continuous.
13. $f : X \to Y$ is called a *homeomorphism* if f is a continuous bijection with a continuous inverse.
14. $f : X \to Y$ is a homeomorphism if and only if $\forall A, f^{-1}(A) \in \mathscr{G}_X \iff A \in \mathscr{G}_Y$.
15. An *embedding* is a continuous injective map $f : X \to Y$ such that $f : X \to f(X)$ is a homeomorphism.
16. An *open neighbourhood* of a point $x \in X$ is a subset $U \in \mathscr{G}$ with $x \in U$. A *neighbourhood* of x is a subset $V \subset X$ such that $x \in U \subset V$ for some open neighbourhood U.
17. A map $f : X \to Y$ is continuous *at a point* $x \in X$ if for any neighbourhood V of $f(x)$, there is a neighbourhood U of x with $f(U) \subset V$.
18. A map $f : X \to Y$ is continuous if and only if it is continuous at every point $x \in X$.

3 Interiors and Closures, Bases and Finite Products

1. (Definitions)
   - **interior**: $A^o := \cup\{U \in \mathscr{G} \mid U \subset A\}$
   - **closure**: $\overline{A} := \cap\{C \in \mathbb{C} \mid A \subset C\}$
   - **boundary**: $\partial A := \overline{A} \backslash A^o$
   - $A \subset X$ is **dense** if $\overline{A} = X$.
   - $A \subset X$ is **nowhere dense** if $\left(\overline{A}\right)^o = \emptyset$
2. (Min/max open set characterization)
   - A^o is the "largest" open subset of A, i.e. $A^o \in \mathscr{G}$, $A^o \subset A$ and $U \in \mathscr{G}, U \subset A \implies U \subset A^o$.
   - $\overline{A}$ is the "smallest" closed subset containing A, i.e. $\overline{A} \in \mathscr{F}$ and $C \in \mathscr{F}$, $A \subset C \implies \overline{A} \subset C$.
   - $A \in \mathscr{G} \iff A^o = A$.
   - $A \in \mathscr{F} \iff \overline{A} = A$.
   - $\partial A \in \mathscr{F}$.
3. (Complement properties)
   - $\overline{A} = (A^o)^c$.
   - $A^o = \overline{A^c}$.
   - $\partial A = \overline{A^c} \cap \overline{A}$
4. (In terms of neighbourhoods)
   - $\overline{A} = \{x \in X \mid \textit{every open neighbourhood } U \textit{ of } x \textit{ has } U \cap A \neq \emptyset\}$.
   - $\partial A = \{x \in X \mid \textit{every open neighbourhood } U \textit{ of } x \textit{ has } U \cap A \neq \emptyset \textit{ and } U \cap A^c \neq \emptyset\}$.
5. A *basis* $\mathscr{B}$ is a subset of $\mathscr{G}$ such that every $U \in \mathscr{G}$ is $\cup_{V \in B} V$ for a suitable $B \subset \mathscr{B}$. Equivalently $\forall U \in \mathscr{G} : \forall x \in U : \exists V \in \mathscr{B} : x \in V \subset U$.
6. If $\mathscr{B}_Y$ is a basis for $\mathscr{G}_Y$, then: $f : X \to Y$ continuous $\iff \forall B \in \mathscr{B}_Y, f^{-1}(B) \in \mathscr{G}_X$.
7. By lecture 1, the open balls $\{B(x, \epsilon)\}_{x \in X, \epsilon > 0}$ form a basis for the metric topology in any metric space.
8. If $\mathscr{B} \subset \mathscr{P}(X)$ satisfies
   - $\forall B_1, B_2 \in \mathscr{B}\ \forall x \in B_1 \cap B_2\ \exists B_3 \in \mathscr{B}\ x \in B_3 \subset B_1 \cap B_2$
   - $\cup \mathscr{B} = X$

   Then $\mathscr{G} = \{\cup B \mid B \subset \mathscr{B}\}$ is a topology on X, and $\mathscr{B}$ is a basis for $\mathscr{G}$. It is called the *topology generated by the basis* $\mathscr{B}$

9. For X, Y topological spaces, $\{U \times V \mid U \in \mathscr{G}_X, V \in \mathscr{G}_Y\}$ satisfies the above properties and generates the so-called product topology on $X \times Y$.

10. If $\mathscr{B}_X$, $\mathscr{B}_Y$ are bases for $\mathscr{G}_X$, $\mathscr{G}_Y$, then the product topology is also generated by $\{U \times V \mid U \in \mathscr{B}_X, V \in \mathscr{B}_Y\}$.

11. For topological spaces, the iterated product topologies on $(X \times Y) \times Z$ and $X \times (Y \times Z)$ coincide, and its basis is
$\{U \times V \times W \mid U \in \mathscr{G}_X, V \in \mathscr{G}_Y, W \in \mathscr{G}_Z\}$.

4 Subbases and General Products

1. If X_1, X_2 are topological spaces, define $\text{pr}_i : (x_1, x_2) \mapsto x_i$, $X_1 \times X_2 \to X_i$, for $i = 1, 2$.
2. (Universal property of the product)
   - pr_i is continuous for $i = 1, 2$.
   - $f : Z \to X \times Y$ is continuous $\iff$ $\text{pr}_i \circ f$ is continuous for $i = 1, 2$.
3. If $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$ are continuous, then $f_1 \times f_2 : y \mapsto (f_1(y), f_2(y))$ is continuous.
4. $\mathscr{S} \subset \mathscr{P}(X)$ is called a *subbasis* of $\mathscr{G}$ if all finite intersections of sets in $\mathscr{S}$ form a basis of $\mathscr{G}$. This implies $\mathscr{S} \subset \mathscr{G}$.
5. If $\mathscr{S}_Y$ is a subbasis of $\mathscr{G}_Y$, then: $f : X \to Y$ continuous $\iff$ $\forall S \in \mathscr{S}_Y, f^{-1}(S) \in \mathscr{G}_X$.
6. $\{\cap_{i=1}^n B_i \mid B_1, ..., B_n \in \mathscr{S}\}$ generates a topology on X if $\mathscr{S}$ is a cover for X. Therefore any cover of $\mathscr{S} \subset \mathscr{P}(X)$ can act as a subbasis of some topology, which is $\{\cup_{i \in I} \cap_{j \in J_i} U_{i,j} \mid U_{i,j} \in \mathscr{S}, J_i \text{ finite}\}$.
7. If $\{\mathscr{G}_\alpha\}_{\alpha \in I}$ is a collection of topologies on X, then $\cap_{\alpha \in I} \mathscr{G}_\alpha$ is a topology on X. It is the finest topology that is coarser than all $\mathscr{G}_\alpha$.
8. The topology generated by its subbasis $\mathscr{S}$ is the coarsest topology that contains S and equals $\cap\{\mathscr{G} \mid \mathscr{G} \text{ topology on } X, \mathscr{S} \subset \mathscr{G}\}$.
9. The product topology on $\prod_{i \in I} X_i$ is the topology generated by $\{\text{pr}_j^{-1}(U_j) \mid j \in I, U_j \in \mathscr{G}_{X_j}\}$.
10. (Universal property of the product)
    - $\forall i \in I, \text{pr}_i$ is continuous.
    - $f : Z \to X \times Y$ is continuous $\iff$ $\forall i \in I, \text{pr}_i \circ f$ is continuous.
11. If $f_i : Y \to X_i$ is continuous for each $i \in I$, then the induced map $f : y \mapsto (f_i(y))_{i \in I}$ is continuous.
12. If $I_1 \cup I_2 = I$ is a partition of I, then $(x_i)_{i \in I} \mapsto ((x_i)_{i \in I_1}, (x_i)_{i \in I_2})$ is a homeomorphism $\prod_{i \in I} X_i \to \prod_{i_1 \in I} X_i \times \prod_{i \in I_2} X_i$
13. If $J \subset I$ is infinite $\forall j \in J, U_j \in \mathscr{G}_{X_j}$, then $\cap_{i \in J} \text{pr}_j^{-1}(U_j)$ is not open in the product topology, (note it contains no basis element).
14. The set $\{\prod_{i \in I} U_i \mid U_i \in \mathscr{G}_{X_i}\}$ generates a (strictly finer) topology, called the box topology
15. If $\mathscr{S}_i$ is a subbasis for $\mathscr{G}_{X_i}$ then $\{\text{pr}_j^{-1}(U_j) \mid j \in I, U_j \in \mathscr{S}_j\}$ is a subbasis for the product topology on $\prod_{j \in I} X_j$.
16. if $f_i : Z \to X_i$ is a family of maps, then the initial topology on Z generated by $\{f_i\}_{i \in I}$ is the set $\{f_i^{-1}(U_i) \mid U_i \in \mathscr{G}_{X_i}\}$. This is the coarsest topology on Z making each f_i continuous.
17. The product topology on $Z = \prod_{i \in I} X_i$ is the initial topology for $\{\text{pr}_i\}_{i \in I}$.

5 Connectivity

1. A topological space X is *connected* if there is no partition of X into two open, nonempty sets.
2. $U \subset X$ is called clopen if $U \in \mathscr{G} \cap \mathscr{F}$.
3. X is connected $\iff$ $\mathscr{F} \cap \mathscr{G} = \emptyset$.
4. $A \subset X$ is *connected* if it is connected when endowed the subspace topology.
5. $A \subset X$ is connected $\iff$ there are no $U, V \in \mathscr{G}$ with $U \cap V \cap A = \emptyset$, $A \subset U \cup V$, $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$.
6. $(0, 1)$ is connected.
7. $A \subset B \subset X$ and $B \subset \overline{A}$ and A connected. Then B is connected.
8. If $f : X \to Y$ is continuous and X is connected then $f(X)$ is connected.
9. If X, Y are homeomorphic, then X is connected $\iff$ Y is connected.
10. All intervals of $\mathbb{R}$ are connected.
11. $\mathbb{Q}$ is disconnected: $\mathbb{Q} \cap (\sqrt{2}, \infty)$, $\mathbb{Q} \cap (-\infty, \sqrt{2})$ is a partition in open disjoint nonempty sets.
12. If $f : X \to \mathbb{R}$ is continuous and $f(x_1) < c < f(x_2)$ then there is a $x \in X$ with $f(x) = c$.
13. If $f : X \to Y$ is continuous, we can define $\varphi : X \to X \times Y$ by $\varphi(x) = \underline{(x, f(x))}$ which is continuous, hence if X connected then $\varphi(X)$ is (the graph graph(f)). $\overline{\text{graph}(f)}$ is also connected.
14. A_i is connected for each $i \in I$ and $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, then $\cup_{i \in I} A_i$ is connected.
15. Define $x \sim y \iff$ there is a connected $A \subset X$ with $x, y \in X$. This is an equivalence relation. Its equivalence classes are called *connected components* of X.
16. (a) If $A \subset [x]$ and A is connected then $[x] = A$.
    (b) $[x] \subset X$ is connected.
    (c) $[x]$ is closed.
17. A connected space has 0 (if $X = \emptyset$) or 1 (if $X \neq \emptyset$) connected components.
18. X is called *totally disconnected* if $\forall x \in X, [x] = \{x\}$.
19. $\mathbb{Q}$ is totally disconnected.
20. A homeomorphism $f : X \to Y$ induces a bijection between the connected components, i.e. $f([x]) = [f(x)]$ is well-defined $X/\sim \to Y/\sim$ and a bijection.
21. A *path* from $x_0 \in X$ to $x_1 \in X$ is a continuous map $w : [0, 1] \to X$ with $w(0) = x_0$, $w(1) = x_1$.
22. A space is called *path connected* if for any $x_0, x_1 \in X$ there is a path from x_0 to x_1.

23. Any path connected space is connected. The converse does not hold: namely $\overline{\mathrm{graph}(f)}$, where f is the topologist's sine curve $f(t) = \sin\frac{1}{t}$, $(0,1] \to [-1,1]$, is connected but not path connected.

24. define $x \sim' y \iff$ there is a path from x to y. This is an equivalence relation on X. The equivalence classes $[x]'$ are called *path components*.

6 Compact Spaces

1. A topological space is called *Hausdorff* or T_2 if $\forall x, y \in X, x \neq y \implies \exists U, V \in \mathscr{G}, U \cap V = \emptyset, x \in U, y \in V$.

2. Any metric space is Hausdorff.

3. Any subspace of a Hausdorff space is Hausdorff.

4. Arbitrary products of Hausdorff spaces are Hausdorff.

5. A topological space is called *quasi-compact* (other texts: *compact*) if $\forall \mathscr{S} \subset \mathscr{G}, \cup \mathscr{S} = X \implies \exists \mathscr{S}' \subset \mathscr{S}$ finite, $\cup \mathscr{S}' = X$.

6. $[0, 1]$ is compact.

7. If X is quasi-compact and $f : X \to Y$ continuous then $f(X)$ is also quasi-compact.

8. If $f : X \to Y$ is a homeomorphism, then X is Hausdorff, quasi-compact or compact then Y also has this property.

9. For $a < b$, $[a, b]$ is not homeomorphic to $\mathbb{R}$ or (a, b).

10. If X is quasi-compact and $F \in \mathscr{F}_X$ then F is compact as a subspace (Because their complement is open, so any open cover induces an open cover $\{U_\alpha\}_{\alpha \in I} \cup \{F^c\}$ with a finite subcover by quasi-compactness of X).

11. If X is Hausdorff, $Y \subset X$ is compact, $x \in X \backslash Y$, then there are disjoint $U, V in \mathscr{G}$ with $x \in U$, $Y \subset V$.

12. If X is Hausdorff and $Y \subset X$ compact, then Y is closed in X.

13. $f : X \to Y$ is called *closed* if $C \in \mathscr{F}_X \implies f(C) \in \mathscr{F}_Y$.

14. $f : X \to Y$ continuous, X quasi-compact and Y Hausdorff. Then f is closed.

15. If $f : X \to Y$ is a continuous bijection with X quasi-compact and Y Hausdorff, then f is a homeomorphism.

16. (Tube Lemma) If X, Y are topological spaces, Y is quasi-compact, $x_0 \in X$, and $W \subset X \times Y$ open with $\{_0\} \times Y \subset W$, then there is an open $U \subset X$ with $U \times Y \subset W$.

17. (Tychonov's Theorem) Arbitrary products of (quasi-)compact spaces are (quasi-)compact.

18. A subset $C \subset \mathbb{R}^n$ is compact $\iff$ C is closed and bounded.

7 Variants of Compactness

1. $s = (x_n)_{n\in\mathbb{N}}$ a sequence in X, $x \in X$. Then x is called an

    - (ω-)**accumulation point** of s if $\forall U \in \mathscr{G}$, $x \in U \implies \{i \in \mathbb{N} \mid x_i \in U\}$ is infinite.
    - a **limit** of s if $\forall U \in \mathscr{G}, x \in U \implies \exists N \in \mathbb{N}\forall n \geq N x_n \in U$.

2. In a Hausdorff space, a sequence has at most one limit.

3. $f : X \to Y$ is continuous, and $(x_n)_{n\geq 1}$ is a sequence in X that has a limit x. Then $f(x)$ is a limit of $(f(x_n))_{n\geq 1}$

4. X is called

    - *first countable* if every $x \in X$ has a countable *neighbourhood base*, i.e. a collection $\{U_n\}_{n\geq 1}$ of neighbourhoods of x such that for every neighbourhood U of x, there is a $n \in \mathbb{N}$ with $U_n \subset U$. Without loss of generality (take $U'_n := \cap m \leq nU_m$), $U_1, \supset U_2 \supset ....$
    - *second countable* if it admits a countable basis.

5. If X is first countable and $f : X \to Y$ is a map such that for any $x \in X$ and any sequence $(x_n)_{n\geq 1}$ with limit x, $f(x)$ is a limit for $(f(x_n))_{n\geq 1}$, then f is continuous.

6. A topological space is

    - *Compact* if it is Hausdorff and quasi-compact.
    - *Countably compact* it is Hausdorff and every countable open cover admits a finite subcover.
    - *Sequentially compact* if every sequence $(x_n)_{x\geq 1}$ admits a convergent subsequence.
    - *Lindelöf* if every open cover admits a countable subcover.

7. A Hausdorff space: is countably compact $\iff$ every sequence $(x_n)_{n\geq 1}$ admits an accumulation point.

8. Lindelöf and countably compact $\implies$ compact.

9. Separable metric spaces are second countable. In fact metrizable spaces are second countable if and only if they are separable.

10. We have:

    - Second countable $\implies$ first countable.
    - Second countable $\implies$ Lindelöf.

11. A Hausdorff space is called

12. A sequentially compact space is countably compact.

13. If X is first countable, Hasudorff, and countably compact, then it is sequenctially compact.

14. We have the following relations:

- Sequentially compact $\implies$ countably compact.
- **First countable**: countably compact $\implies$ sequentially compact
- Compact $\implies$ countably compact.
- **Second countable**: countably compact $\implies$ sequentially compact

Hence, in second countable Hausdorff spaces, all three variants of compactness are equivalent.

15. A countably compact metric space (X, d) is second countable, therefore a metric space is compact $\iff$ countably compact $\iff$ sequentially compact.
16. A metric space is called *totally bounded* if for any choice of $\epsilon > 0$, X can be covered with finitely many ϵ-balls.
17. A metric space is compact $\iff$ it is totally bounded and complete.

8 The Cantor Set and the Peano Curve

1. Every $x \in [0,1]$ admits a ternary expansion $x = \sum_{i=0}^{\infty} \frac{a_i}{3^i}$, $a_i \in \{0,1,2\}$. Such an expansion is not unique: $\sum_{i=m}^{\infty} \frac{2}{3^i} = \frac{1}{3^{m-1}}$.

2. The *Cantor set* is $\cap_{i=1}^{\infty} C_i$, where $C_0 = 1$ and C_i is obtained from C_{i-1} by deleting the middle third open interval from each consecutive interval of C_{i-1}. C_n consists of 2^n closed intervals of length 3^{-n}. Alternatively, $C = \{\sum_{i=1}^{\infty} \frac{a_i}{3^i} \mid a_i \in \{0,2\}, \forall i\}$. The Cantor set here inherits the subspace topology from the Euclidean topology on $[0,1]$.

3. (Properties of the Cantor set)
   - C is compact.
   - C does not contain any open interval (a,b) with $a < b$.
   - $C^o = \emptyset$, C is nowhere dense, and C is totally disconnected.

4. For elements $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, $y = \sum_{i=1}^{\infty} \frac{b_i}{3^i}$ with $\forall i,\ a_i, b_i \in \{0,2\}$:
   - If $\forall 1 \le i \le m\ a_i = b_i$, then $|x-y| \le 3^{-m}$.
   - If additionally $a_{m+1} \ne b_{m+1}$, then $|x-y| \ge 3^{-(m+1)}$.
   - If $|x-y| < 3^{-m}$, then $\forall 1 \le i \le m\ a_i = b_i$
   - If $x = y$ then $\forall i \ge 1\ a_i = b_i$

5. For $A \subset X$, X a topological space, $x \in X$ is called
   - a *limit point* of A if every neighbourhood U of x has at least one $y \in U \cap A$ with $y \ne x$;
   - an *isolated point* if there is a neighbourhood U of x with $U \cap A = \{x_0\}$.

   We see x is an isolated point of $A \iff \{x\}$ is open relative to A.

6. Every $x \in C$ is a limit point of C.

7. $C \to \{0,2\}^{\mathbb{N}}$ by $\sum_{i=1}^{\infty} \frac{a_i}{3^i} \mapsto (a_i)_{i=1}^{\infty}$ is well-defined by 4. and a homeomorphism with respect to the subspace topology on C and the product topology on $\{0,2\}^{\mathbb{N}}$.

8. There is a homeomorphism $C \to C \times C$.

9. (Differences between C and $I = \mathbb{Q} \cap [0,1]$)
   - C is uncountable, I is countable.
   - C is nowhere dense, I is dense in $[0,1]$.
   - C is compact, I is not closed.
   - C and I both have empty interior and no isolated points, and Lebesgue measure 0.

10. The map $C \to [0,1]$, $\sum_{i=1}^{\infty} \frac{a_i}{3^i} \mapsto sum_{i=1}^{\infty} \frac{a_i}{2} \frac{1}{2^i}$ is continuous and surjective.

11. There is a continuous surjection $f : [0,1] \to [0,1] \times [0,1]$ with $f|_C : C \to C \times C$ the homeomorphism mentioned in 8..

12. For each $n \ge 1$, there is a homeomorphism $\mathbb{R} \to \mathbb{R}^n$.

9 Quotient Spaces

1. If $f : X \to Y$ is surjective, X a topological space and Y a set:
   - $\mathscr{G}_f = \{V \subset Y \mid f^{-1} \in \mathscr{G}_X\}$ is a topology on Y.
   - f is continuous with respect to this topology.
   - $\mathscr{G}_f$ is the finest topology on Y for which f is continuous.
2. If $(f^{-1}(V) \in \mathscr{G}_X \implies V \in \mathscr{G}_Y)$ and f is continuous and surjective, then f is called an *identification*. This means precisely $\mathscr{G}_Y = \mathscr{G}_f$.
3. The composition of two identifications is an identification.
4. If f is bijective, then f is an identification $\iff$ f is a homeomorphism.
5. If $f : X \to Y$ is an identification and $h : X \to Z$ is continuous such that $\forall x_0, x_1 \in X : f(x_0) = f(x_1) \implies h(x_0) = h(x_1)$, then there is a unique $g : Y \to Z$ with $gf = h$. Moreover, g is continuous.
6. If X is a topological space with equivalence relation $\sim \subset X^2$, then we define a topology on $X/\sim$ as $\mathscr{G}_q$ where $q : x \mapsto [x]$ is the quotient map. q is by definition an identification.
7. The arbitrary intersection $\cap_{\sim \in \mathscr{S}} \sim$ of equivalence relations, is an equivalence relation. For $A \subset X$, one can define a smallest equivalence relation $\sim_A$ such that $A \times A \subset \sim_A$ (the equivalence relation *generated by* A), and define $X/A := X/\sim_A$.
8. If X is connected, path connected or quasi-compact, so is $X/\sim$. Taking quotients does generally not preserve the Hausdorff property: consider $[0,2]/[0,1)$. Any open neighbourhood of $[1]$ contains $[0]$.
9.
   - $D^n = \{x \in \mathbb{R}^n \mid |x|_2 \leq 1\}$.
   - $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|_2 = 1\} = \delta D^{n+1}$.

   D^n, S^n are compact for $n \geq 0$.
10.
    - $[0,1]/\{0,1\} \cong S^1$.
    - $\frac{S^{n-1} \times [0,1]}{S^{n-1} \times \{1\}} \cong D^n$.
    - $\frac{D^n}{\partial D^n} \cong S^n$.
11. (*Torus*) Let:
    - $(s, 0) \sim (s, 1)\ \forall s \in [0,1]$.
    - $(0, t) \sim (1, t)\ \forall t \in [0,1]$.

    Then $[0,1]^2/\sim \cong S^1 \times S^1$.
12. (*Möbius Band*) Let:
    - $(0, t) \sim (1, 1-t)\ \forall t \in [0,1]$.

Then $[0,1]^2/\sim$ is called the *Möbius band*.

13. (*Klein Bottle*) Let:

    - $(s,0) \sim (s,1)$ $\forall s \in [0,1]$.
    - $(0,t) \sim (1,1-t)$ $\forall t \in [0,1]$.

    Then $[0,1]^2/\sim$ is called the *Klein bottle*.

10 Separation Axioms

1. A topological space is called
   - T_1 if $\forall x, y \in X, x \neq y \implies \exists U \in \mathscr{G}, x \in U, y \notin U$.
   - T_2 or *Hausdorff* if $\forall x, y \in X : x \neq y \implies \exists U, V \in \mathscr{G}, U \cap V = \emptyset, x \in U, y \in V$.
   - T_3 if $\forall C \in \mathscr{F}, x \in C^c : \exists U, V \in T : C \subset U, x \in V, V \cap U = \emptyset$.
   - T_4 if $\forall C_0, C_1 \in \mathscr{F} : C_0 \cap C_1 = \emptyset \implies \exists U, V \in T : C_0 \subset U, C_1 \subset V, V \cap U = \emptyset$.
2. - T_1 is equivalent to: $\forall x \in X, \{x\}$ is closed.
   - T_3 is equivalent to: $\forall x \in X : \forall U$ open neighbourhood of x: $\exists C$ closed neighbourhood of x with $C \subset U$.
   - T_4 is equivalent to: $\forall C \in \mathscr{F}$, and all U open neighbourhood of C, there is an open neighbourhood V of C with $\overline{V} \subset U$.
   - (*Urysohn's Lemma, Lecture 11*)
     T_4 is equivalent to: $\forall C_0, C_1 \subset X$ closed nonempty disjoint subsets, there is a $f : X \to [0,1]$ with $f(C_0) = \{0\}$, $f(C_1) = \{1\}$.
3. A topological space is called
   - *Regular* if it is T_1 and T_3.
   - *Normal* if it is T_1 and T_4.
4. The definition is easy to check, but we have:
   Normal $(T_1 + T_4) \implies$ Regular $(T_1 + T_3) \implies$ Hausdorff $(T_2) \implies T_1$.
5. Any compact space is normal.
6. A second countable regular space X is metrizable (there is a metric $d : X^2 \to [0, \infty))$ such that $\mathscr{G}_X = \mathscr{G}_d$) (proof: use Urysohn's lemma to construct an embedding $X \to [0,1]^{\mathbb{N}}$ and metrize the latter space $d(x,y) := \sum_{i=1}^{\infty} |x_i - y_i| 2^{-i} (1 + |x_i - y_i|)^{-1}$.
7. If X is a topological space and $Y \subset X$, and X a T_k-space for $k = 1, 2$ or 3, then Y is also a T_k-space.
8. If X is a topological space and $Y \subset X$ and Y is closed, and X is T_4, then Y is a T_4-space (this is because $\mathscr{F}_Y \subset \mathscr{F}_X$).
9. If $\{X_i\}_{i \in I}$ is a collection of topological spaces, $k \in \{1, 2, 3\}$. Then $\prod_{i \in I} X_i$ is $T_k \iff \forall i \in I$, X_i is T_k.
10. There is no such statement for general products of T_4 spaces.
11. For $\{X_i\}_{i \in I}$ a collection of topological spaces,
    - $\coprod_{i \in I} X_i = \{(i, x) \mid i \in I, x \in X_i\}$
    - $\text{incl}_i : X_i \to \coprod_{i \in I} X_i$ is $x \mapsto (i, x_i)$
    - $A \subset \coprod_{i \in I} X_i$ is called *open* if $\forall i \in I$, $\text{incl}_i^{-1}(A)$ is open in X_i.

This defines the *coproduct topology*.

12. The quotient topology $\mathscr{G}_q$ and the coproduct topology are a particular case of a *final topology*: given a collection $\{f_i : X_i \to Y\}_{i\in I}$ this is the finest topology on Y such that each f_i is continuous.

13. If $k \in \{1,2,3,4\}$, then $\coprod_{i\in I} X_i$ is $T_k \iff \forall i \in I$, X_i is T_k

14. A topological space X is disconnected $\iff$ X is homeomorphic to a coproduct of two nonempty spaces.
    Note that X is not homeomorphic to the disjoint union of its connected components.

15. Passing to a quotient space may destroy all separation properties: $X = [0,1]$ is compact and therefore T_k for $k = 1,2,3,4$. Take $A = X \cap \mathbb{Q}$, then X/A.

    - $\{q(0)\} \in X/A$ is not closed since $q^{-1}(\{q(0)\}) = X/A$ is not. So X/A is not T_1.
    - Therefore, X/A is not Hausdorff.
    - For $x \in X$ irrational, $\{q(x)\} \subset A$ is closed since $q^{-1}((X/A)\backslash\{q(x)\}) = X\backslash\{x\}$ is open. Any neighbourhood of x contains a rational number, so any neighbourhood of $\{q(x)\}$ contains $q(0)$. Hence X/A is not T_3.
    - If $x, y \in X$ are distinct and irrational, the same argument shows any pair of open neighbourhoods of $q(x)$ and $q(y)$ intersect at $q(0)$. Hence X/A is not T_4.

16. For $f : X \to Y$, a set $A \subset X$ is called *saturated* if $f^{-1}(f(A)) = A$ holds. If f is an identification, $f(U)$ is open in Y if U is saturated and open.

17. If $f : X \to Y$ is a closed identification, $C \subset X$ saturated and closed and $U \subset X$ a closed neighbourhood of C. Then there is a saturated open neighbourhood V of C with $C \subset V \subset U$.

18. Let X be normal and $f : X \to Y$ be a closed identification. Then Y is normal.

19. If X is compact and $f : X \to Y$ is an identification, then f is closed $\iff$ Y is Hausdorff.

11 Extension Theorems

1. $d(x, A) := \inf_{y \in A} d(x, y)$, $d(\cdot, A) : X \to [0, \infty)$.

2. $|d(x, A) - d(y, A)| \leq d(x, y)$, so $d(\cdot, A)$ is Lipschitz continuous. If $A \in \mathscr{F}$, then $d(x, A) = 0 \iff x \in A$.

3. (X, d) compact metric space, $\{U_i\}_{i \in I}$ open cover of X, then there is an $\epsilon > 0$ with $\forall i \in I : \forall x \in X : B(x, \epsilon) \subset U_i$

4. (X, d) metric space and $C_0, C_1 \in \mathscr{F}$, $C_0 \neq \emptyset$, $C_1 \neq \emptyset$, $C_0 \cap C_1 = \emptyset$. Then there is a continuous $f : X \to [0, 1]$ with $f(C_0) = \{0\}$, $f(C_1) = \{1\}$.

5. X is $T_4 \iff \forall C \in \mathscr{F}$, and all U open neighbourhood of C, there is an open neighbourhood V of C with $\overline{V} \subset U$.

6. (Urysohn's Lemma) If X is a topological space, then X is $T_4 \iff$ for all $C_0, C_1 \subset X$ closed nonempty disjoint subsets, there is a $f : X \to [0, 1]$ with $f(C_0) = \{0\}$, $f(C_1) = \{1\}$.

7. Metric spaces are normal.

8. (Tietze's Extension Theorem) If X is T_4 and $C \subset X$ closed, then for every continuous $g : C \to [0, 1]$ there is a continuous $f : X \to [0, 1]$ with $f|_C = g$.

9. This implies, immediately, the existence of a Peano curve based on the continous surjection $C \to [0, 1]$, $\sum_{i=1}^{\infty} a_i 3^{-1} \mapsto \sum_{i=1}^{\infty} a_i 2^{-i-1}$.

10. A topological space X is called *locally compact* if it is Hausdorff and every neighbourhood admits a compactt neighbourhood.

11. Compact spaces are locally compact. $\mathbb{R}^n$ is locally compact for $n \geq 1$.

12. If X is locally compact, $U \subset X$ a neighbourhood of $x \in X$, then U contains a compact neighbourhood of x.

13. Locally compact spaces are regular (because, any neighbourhood $U \ni x \in X$ contains a compact neighbourhood K, and K is compact, so is closed since X is Hausdorff, and this means any neighbourhood of any $x \in X$ contains a closed neighbourhood, which is equivalent to T_3. Hausdorff implies T_1).

14. If X is locally compact and $U \subset X$ is open, then U is locally compact in the subspace topology.

15. (One-point compactification topology) For X, write $X^+ = X \cup \{P\}$ with $P \notin X$. Define $U \subset X^+$ to be open if

    - $U \subset X$ and U is open in X.
    - $P \in U$ and $X^+ \backslash U$ is compact in X.

    This forms a topology on X^+ such that X^+ is compact and $X \mapsto X^+$ is an embedding.

16. Any $f : X \to Y$ extends to $f^+ : X^+ \to Y^+$ with $f^+|_X = f$ and $f(P_X) = P_Y$.

- If f is continuous, and additionally if $K \subset Y$ is compact hen $f^{-1}(K) \subset X$ is compact, f^+ is continuous.
- If X, Y, are locally compact and f is a homeomorphism, then f^+ is a homeomorphism.

17. If X is compact and $x_0 \in X$, then since X is T_1, $X \backslash \{x_0\}$ is open and therefore locally compact as an open subspace of X locally compact. $f : (X \backslash \{x_0\})^+ \to X$ which sends P to x_0 and is the identity on $X \backslash \{x_0\}$ is a homeomorphism

18. For $e_{n+1} = (\overbrace{0, ..., 0}^{\times n}, 1) \in \mathbb{R}^{n+1}$ the *north pole* of $S^n \subset \mathbb{R}^{n+1}$, define the *stereographic projection* and its inverse as:

$$h : S^n \backslash \{e_{n+1}\} \to \mathbb{R}^n, \quad (x_0, ..., x_n) \mapsto \frac{1}{1 - x_n}(x_0, ..., x_{n-1})$$
$$h^{-1} : \mathbb{R}^n \to S^n \backslash \{e_{n+1}\}, \quad (y_1, ..., y_n) \mapsto \frac{1}{|y|^2 + 1}(2y_1, ..., 2y_n, |y|^2 - 1)$$

h induces a homeomorphism $(\mathbb{R}^n)^+ \cong S^n$.

12 The Fundamental Group

- - A pointed space (X, x_0) is a topological space $(X, \mathscr{G})$ with $x_0 \in X$.
  - A *loop* in X with basepoint $x_0 \in X$ is a path $w : [0,1] \to X$ with $w(0) = w(1) = x_0$
  - A *homotopy of loops* from w_0 to w_1 (loops with basepoint $x_)$) is a continuous $H : [0,1]^2 \to X$ with
    - $H(s,i) = w_i(s)$ for $i = 0, 1,\ s \in [0,1]$.
    - $H(i,t) = x_0$ for $i = 0, 1,\ t \in [0,1]$.
  - Two loops w_0, w_1 are *homotopic* if there exists an endpoint-preserving homotopy of loops, denoted $w_0 \simeq_{x_0} w_1$. This is an equivalence relation on all loops with basepoint x_0. The equivalence class of w is denoted $[w]$

- The *fundamental group* of (X, x_0) is denoted $\pi_1(X, x_0) = X/ \simeq_{x_0} = \{[w] \mid w \text{ loop with basepoint } x_0\}$.

  It is a group under $[w] \cdot [v] = [w * v]$ where $*$ denotes concatenation of paths:

  - $w * v(t) = \begin{cases} w(2t) & t \in [0, \frac{1}{2}] \\ v(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$
  - The inverse $[w]^{-1} = [\overline{w}]$ where $\overline{w}(t) = w(1-t)$.
  - One needs to show that this is well-defined:
    $v_0 \simeq_{x_0} v_1,\ w_0 \simeq_{x_0} w_1 \implies v_0 * w_0 \simeq_{x_0} v_1 * w_1$.
  - In order to show associativity, $(u * v) * w \simeq_{x_0} u * (v * w)$
  - In order to show the identity relation, $\text{const}_{x_0} * v \simeq_{x_0} v$
  - In order to show the inverse relation, $\overline{w} * w \simeq_{x_0} \text{const}_{x_0} * v \simeq_{x_0} w * \overline{w}$.

- For every $n \geq 1$ and every $x_0 \in \mathbb{R}^n$, $\pi_1(\mathbb{R}^n, x_0) = \{[\text{const}_{x_0}]\}$

- For $f_0, f_1 : X \to Y$ continuous, a *homotopy* from f_0 to f_1 is a continuous $H : X \times [0,1] \to Y$ with $H(t,i) = f_i(t)\ i = 1, 2,\ t \in [0,1]$. This is an equivalence relation on $C(X,Y)$.

- A homotopy of loops is a homotopy of maps $w_0, w_1 : [0,1] = X \to Y$ with the extra condition that $H|_{\{0,1\} \times [0,1]} = \text{const}_{x_0}$.

- If $f : X \to Y$ is continuous and $x_0 \in X$, then f induces a well-defined map of sets

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)), \quad [w] \mapsto [f \circ w]$$

- Properties of f_*:

  - f is a group homomorphism: $[f \circ (v * w)] = [(f \circ v) * (f \circ w)]$.
  - If $f : X \to Y$, $g : Y \to Z$ are continuous, $(g \circ f)_* = g_* \circ f_*$
  - $(\text{id}_X)_* = \text{id}_{\pi_1(X, x_0)}$.
  - If f is a homeomorphism, f_* is a group isomorphism. Its inverse is $(f_*)^{-1} = (f^{-1})_*$
  - This can be generalised: f_* is an isomorphism already if $f : X \to Y$ is a homotopy equivalence with $g : Y \to X$ continuous s.t. $gf \simeq \text{id}_X$, $fg \simeq \text{id}_Y$, then $(f_*)^{-1} = g_*$. (see Lecture 13).

13 Fundamental Groups and Homotopy Equivalences

- If X is a topological space and v a loop based at x_0. Then v induces a group isomorphism

$$v_* : \pi_1(X, v(0)) \to \pi_1(X, v(1)), \quad v_*([w]) = [(\overline{v} * w) * v]$$

  Note the explicit parentheses: this is because the concatenation done is not of loops but of paths, so there is no associativity shown yet. Yet one can show $(\overline{v} * w) * v \simeq_{v(0)} \overline{v} * (w * v)$.

- The above shows that $\pi_1(X, x_0)$ is, up to isomorphism, invariant under continuous translation of the basepoint.

- If $f : X \to Y$ is continuous and $v : [0,1] \to X$ a path, $f_* \circ v_* = (f \circ v)_* \circ f_*$

- $f : X \to Y$ continuous is called a *homotopy equivalence* if there is a continuous $g : Y \to X$ with $gf \simeq \mathrm{id}_X$, $fg \simeq \mathrm{id}_Y$.

- Any homeomorphism is a homotopy equivalence.

- If X is a topological space and $H : [0,1]^2 \to X$ continuous with $H(0,0) = H(1,0)$, then in $\pi_1(X, H(0,0))$, it holds:

$$[H|_{[0,1]\times\{0\}}] = [(H_{\{0\}\times[0,1]} * H|_{[0,1]\times\{1\}} * \overline{H|_{\{1\}\times[0,1]}}]$$

- If $f : X \to Y$ is a homotopy equivalence with $g : Y \to X$ continuous s.t. $gf \simeq \mathrm{id}_X$, $fg \simeq \mathrm{id}_Y$, then $(f_*)^{-1} = g_*$ and f_* is a group isomorphism.

- Two spaces X, Y are called *homotopy equivalent* if there is a homotopy equivalence $f : X \to Y$ (equivalence relation).

- A space X is called *contractible* if it is homotopy equivalent to a one point space $\iff$ there is a $x_0 \in X$ with $\mathrm{incl}_{\{x_0\}} : x_0 \mapsto x_0$ is a homotopy equivalence.

- $A \subset X$ is called a *deformation retract* if there is a continuous $r : X \to A$ with $r|_A = \mathrm{id}_A$ and a homotopy $H : X \times [0,1] \to X$ from id_X to $\mathrm{incl}_A \circ r$ such that $H(x,t) = x$ for all t and $x \in A$.

- If $A \subset X$ is a deformation retract, then $\mathrm{incl}_A : A \to X$ is a homotopy equivalence.

- The converse is not true: the comb space

$$X = \{0\} \times [0,1] \cup [0,1] \times \{0\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{Z}_{\geq 0}\} \times [0,1]$$

  is contractible to $x_0 = (0,0)$, but not deformation retractible to $y_0 = (0,1)$, for example. If there is a homotopy, that is a continuous map $H : X \times [0,1] \to X$ with

$$H(\cdot, 0) = \mathrm{id}, \; H(\{y_0\} \times [0,1]) = \{y_0\}, \text{ and } H(X \times \{1\}) = \{y_0\}$$

  Then take the sequence $x_n = (2^{-n}, 1)$ (the tips of the comb's teeth), $x_n \to y_0$, and since $H(x_n, \cdot)$ is a path in X, let t_n be the point at which $H(x_n, t_n) = (2^{-n}, 0)$ (the point x_n has reached the base). Then by $[0,1]$ compact, hence sequentially compact, pick a subsequence $t_{n_k} \to \hat{t}$, then $x_{n_k} \to y_0$ by convergence of x_n, so by continuity of H we get

$$H(x_{n_k}, t_{n_k}) \to H(y_0, \hat{t}) = y_0$$

While by choice of t_n such that $H(x_n, t_n) = (2^{-n}, 0)$, we get

$$H(x_{n_k}, t_{n_k}) = (2^{n_k}, 0) \to x_0$$

Contradiction, since X is Hausdorff. This means any homotopy equivalence $\mathrm{incl}_{\{y_0\}} \circ r \simeq \mathrm{id}_X$ will move some point $x \in A$. Note that X still contracts to y_0, because it contracts to x_0 and then we use a path from x_0 to y_0 to construct a homotopy from $\mathrm{incl}_{\{x_0\}}$ to $\mathrm{incl}_{\{y_0\}}$.

- - $\mathbb{R}^{n+1} \backslash \{0\}$ has S^n as a deformation retract. Homotopy: $H(x,t) = (1-t)\frac{x}{|x|} + tx$.
  - D^n and $\mathbb{R}^n$ have $\{0\}$ as a deformation retract. Homotopy: $H(x,t) = (1-t)x$.
- For $n \geq 2$ and $s_0 \in S^n$, $\pi_1(S^n), s_0)$ is trivial.
- Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and define $\exp : x \mapsto e^{2\pi i x}$, $\mathbb{R} \to S^1$, let $\varphi_n : [0,1] \to S^1$, $\varphi_n(t) = \exp(nt) = \exp(t)^n$. Then $\phi(n) = [\varphi_n]$ is a group isomorphism $\mathbb{Z} \cong \pi_1(S^1, 1)$.

  The proof requires theory developed in Lecture 14.
- (*Drum theorem*): There is no continuous map $r : D^2 \to \partial D^2$ with $r|\partial D^2 = \mathrm{id}_{\partial D^2}$.

14 Fundamental Groups and Covering Spaces

- Preliminaries:
  - $\coprod_{i\in I} X \cong I \times X$, where I has the discrete topology and $I \times X$ has the product topology induced by I and X.
  - If X is a topological space and $V_i \subset X$, if all V_i are disjoint and open, then $\cup_{i\in I} V_i \cong \coprod_{i\in I} V_i$.
- For any $z \in S^1$, there is an open $U_z \subset S^1$, $z \in U_z$ and a homeomorphism $h : U \times \exp^{-1}(\{z\}) \to \exp^{-1}(U)$ where $\exp^{-1}(U) \subset \mathbb{R}$ has subspace topology and $\exp^{-1}(\{z\}) = \{\in \mathbb{R} \mid \exp(x) = z\}$ has the discrete topology. In other words, $\exp^{-1}(U) \cong U^{\mathbb{Z}}$
- S^1 admits an open cover such $\{U_j\}_{j\in J}$ that for each $y \in \mathbb{R}$ an each $j \in J$ with $\exp(y) \in U_j$, there is an open $V \subset \mathbb{R}$ such that $\exp|_V : V \to U_j$ is a homeomorphism.
- (*Homotopy lifting property of* exp) If $F : [0,1] \times [0,1] \to S^1$, $g : [0,1] \to \mathbb{R}$ continuous with $\exp \circ g(s) = F(s,0)\ \forall s \in [0,1]$. Then there is a unique continuous $G : [0,1] \times [0,1] \to \mathbb{R}$ with $\exp \circ G = F$ and $G(s,0) = g(s)\ \forall s \in [0,1]$.

  This means that local invertibility of exp is sufficient to lift F to G (if exp were invertible, we could trivially take $G = \exp^{-1} \circ F$).
- (*Path lifitng property of* exp) For $[0,1] \to S^1$ path and $x \in \mathbb{R}$ with $\exp(x) = w(0)$, there is a unique path $v : [0,1] \to \mathbb{R}$ with $\exp \circ v = w$ and $v(0) = x$.
- $\phi(n) = [\varphi_n]$ is a group isomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$.
- For B, E topological spaces, E is called a *covering space* for B if there is a surjective map $p : E \to B$ with $\forall x \in B \exists U_x \in \mathscr{G}_B : \exists h : U \times p^{-1}(\{x\}) \to p^{-1}(U) : h$ homeomorphism and $p \circ h = \mathrm{pr}_U$. Here $U \times p^{-1}(\{x\})$ carries the product topology, where $p^{-1}(\{x\})$ carries the discrete topology. p is called a *covering map*.
- X is called *simply connected* if X is path connected and $\pi_1(X, x_0)$ is trivial for one (hence any) $x_0 \in X$.
- A *universal covering* of a topological space B is a covering map $p : E \to B$ with E simply connected.
- exp is a universal covering $\mathbb{R} \to S^1$.
- If B is path connected and any neighbourhood of a (hence any) point $x \in B$ contains a path connected neighbourhood U of x such that the induced map $\pi_1(U, x) \to \pi_1(B, x)$ is trivial. Then there is a universal covering $p : E \to B$.
- For p a covering map, a *deck transformation* $f : E \to E$ is a homeomorphism such that $p \circ f = p$. The set of all deck transformations is denoted $\mathrm{Aut}(p)$ and is a group under function composition.
- If $p : E \to B$ is a universal cover of a path connected space B then for any $x_0 \in B$, $\pi_1(B, x_0) \cong \mathrm{Aut}(p)$.