Optimization in Geometry & Physics

1.1

Facts (Preliminaries)

$C^0([a,b])$ is complete with C^0 norm

$C^1([a,b])$ is complete with C^1 norm
$$|y|_1 = |y|_0 + |y'|$$

$C^{1pw}([a,b])$ is not complete with $C^{1,pw}$ norm *

Our "functionals" will be functions

$$J : C^{x}([a,b]) \longrightarrow \mathbb{R}$$

where x can by any modification
Functionals can be viewed as maps between vector spaces, and in particular the norm induces a topology in which we can consider sequential continuity.

def Functional $J : D \to \mathbb{R}$ is called cont. at y if

$$\forall \varepsilon > 0 \ \exists \delta > 0$$
$$\forall_{\tilde{y} \in B(y,\delta)} \ |J(\tilde{y}) - J(y)| < \varepsilon$$

if $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous
then
$$J(y) := \int_a^b F(x, y(x), y'(x))\, dx$$

is continuous $(C^{1,pw}, |\cdot|_{1,pw}) \longrightarrow (\mathbb{R}, |\cdot|)$

* Kielhöfer, exercise 1.1.1

1.2

def 1.2.1 $\mathcal{J}: D \to \mathbb{R}$ functional on $D \subset X$, X normed space.

If D is open, we have

$$\forall h \in X, \quad \exists \varepsilon > 0, \quad \forall t : |t| < \varepsilon, \quad y + th \in D$$

therefore we can $\forall h \in X$ define a function $g : (-\varepsilon_h, \varepsilon_h) \to \mathbb{R}$ as

$$g(t) = \mathcal{J}(y+th)$$

then we define $d\mathcal{J}(y,h) = g'(0)$

$$= \lim_{t \to 0} \frac{\mathcal{J}(y+th) - \mathcal{J}(y)}{t}$$

Provided it exists.

Fact: if existent, then $d\mathcal{J}(y, \alpha h)$ also exists and $d\mathcal{J}(y, \alpha h) = \alpha\, d(\mathcal{J}, h)$

but in general, $h \mapsto d\mathcal{J}(y,h)$ need not be additive, hence is not necessarily linear.

Already in $\dim(X) < \infty$ there are counterex.

such as

$$\mathcal{J}: \mathbb{R}^2 \to \mathbb{R} \quad \mathcal{J}(y) = \begin{cases} y_1^2 \left(1 + \frac{1}{y_2}\right) & y_2 \neq 0 \\ 0 & y_2 = 0 \end{cases}$$

$\Rightarrow$ $h_2 \neq 0$ gives $\displaystyle\lim_{t \to 0} \frac{\mathcal{J}(y+th) - \mathcal{J}(y)}{t} = \lim_{t \to 0} \frac{t^2 h_1^2}{t} + \frac{t^2 h_1^2}{t^2 h_2} = \frac{h_1^2}{h_2}$

$h_2 = 0$ gives $\displaystyle\lim_{t \to 0} \frac{\mathcal{J}(0+th) - \mathcal{J}(0)}{t} = \lim_{t \to 0} \frac{t^2 h_1^2}{t} = 0$

X normed space
$\cup$

def Fréchet diffable at $y \in D$: $J : D \to \mathbb{R}$ is called this if there is a bounded linear operator * $L : X \to \mathbb{R}$

s.t $$\lim_{\|h\|_X \to 0} \frac{\| J(y+h) - J(y) - L(h)\|_Y}{\|h\|_X} = 0$$

* bounded: it maps bounded $S \subset X$ to bounded $L(S) \subset \mathbb{R}$. $\Leftrightarrow$ $\exists R > 0$:
$\forall x \in X : \|T(x)\|_Y \leq R\|x\|_X$

In $X \cong \mathbb{R}^n$, this is simply total diff.ability
It implies linearity of $dJ(y,h)$ as:

$$\frac{d}{dt} J(y+th) = \partial_h J(y) = \langle \nabla J(y), h \rangle$$

def If $dJ(y,h)$ exists in $D \subset X$, $h \in X$ and is linear in h, we call it the first variation of J in dir. h

denoted $\delta J(y) h$

defines lin.op $\delta J(y) : X \to \mathbb{R}$

we often only care for linearity on a subspace $h \in X_0 \subset X$. Then $\delta J(y) : X_0 \to \mathbb{R}$ is the first variation.

remark Fact: on X fin.dim, any L linear op. is continuous as its matrix is finite so has a bounded norm
Not the case for $\dim(X) \notin \mathbb{N}$.

for example, $X = C^1([0,1])$, $T : X \to \mathbb{R}$, with X having C^0 norm and define

$$T(y) = y'(1).$$

Take $y_n(x) = \frac{1}{n} x^n$. Then $y_n \xrightarrow{C^0} 0$

But $y_n'(1) = x^{n-1}\big|_{x=1} = 1 \to 1 \neq T(0)$

$\square$

prop 1.2.1

$$\mathcal{J}(y) = \int_a^b F(x, y(x), y'(x))\, dx$$

defined from $F : [a,b] \times \mathbb{R} \times \mathbb{R}$

and $D \subset C^{1,pw}([a,b])$.

Assume

1) F is continuous and cont diff.able in variables y and y'

2) $\begin{cases} \forall y \in D \\ \forall h \in C_0^{1,pw}([a,b]) \quad \exists \varepsilon > 0 \quad y + th \in D \end{cases}$

Then $\mathcal{J}$ has a first variation $\delta \mathcal{J}(y)h$ $\forall h \in C_0^{1,pw}([a,b])$ namely

$$\delta \mathcal{J}(y) h = \int_a^b \left(F_y(x,y,y') h + F_{y'}(x,y,y') h' \right) dx$$

1.2.1 $$J(y) = \int_a^b F(x, y(x), y'(x))\, dx$$

defined on $D \subset C^{1,pw}[a,b]$

and $\forall y \in D$ $\forall h \in C_0^{1,pw}[a,b]$ $\exists \varepsilon > 0$ $\forall t \in (-\varepsilon, \varepsilon)$ $y + th \in D$

and F is continuous and F_y, $F_{y'}$ exist

and are continuous.

Then $$dJ(y,h) = \int_a^b \Big(F_y(x, y(x), y'(x))\, h(x) + F_{y'}(x, y(x), y'(x))\, h'(x) \Big) dx$$

exists and is clearly linear in h,

therefore $\delta J(y) h$ exists $(\forall y \in D\ \forall h \in C_0^{1pw})$

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proof

fix $y \in D$ fix $h \in C^{1pw}$ and let ε be as in the premise. fix x.

$\forall t \in (-\varepsilon, \varepsilon) \setminus \{0\}$, we then have:

$$\frac{1}{t}\Big(F(x, y(x) + th(x), y'(x) + th'(x)) - F(x, y(x), y'(x)) \Big)$$

$$= \frac{1}{t} \int_0^t \frac{d}{ds}\Big(F(x, y(x) + sh(x), y'(x) + sh'(x)) \Big) ds$$

$\uparrow$ valid since $s \mapsto F(x, y+sh, y'+sh')$ is diff'able due to chain rule)

[abbreviate $y = y(x)$, $y' = y'(x)$, $h = h(x)$, $h' = h'(x)$

$$= F_y(x,y,y')h + F_{y'}(x,y,y')h'$$
$$+ \frac{1}{t} \int_0^t \underbrace{\big(F_y(x, y+sh, y'+sh') - F_y(x,y,y')\big)h}_{U}\, ds$$
$$+ \frac{1}{t} \int_0^t \underbrace{\big(F_{y'}(x, y+sh, y'+sh') - F_{y'}(x,y,y')\big)h'}_{V}\, ds$$

= [mean-value theorem for integral]

$$F_y(x,y,y') + F_{y'}(x,y,y') + U(t_0) + V(t_1)$$

where $t_0, t_1 \in [0,t]$ and

$U(\tau) := [F_y(x, y+\tau h, y'+\tau h') - F_y(x,y,y')]\, h(x)$ both

$V(\tau) := [F_{y'}(x, y+\tau h, y'+\tau h') - F_{y'}(x,y,y')]\, h'(x)$ continuous at $\tau = 0$

$\Rightarrow$ as $t \to 0$, $t_0, t_1 \to 0$ so $U(t_0) \to U(0) = 0$

$V(t_1) \to V(0) = 0$

$\Rightarrow$ as $t \to 0$, $\frac{1}{t}\big[F(x, y+th, y'+th') - F(x,y,y')\big] \to$

$F_y(x,y,y')h + F_{y'}(x,y,y')h'$

now, as the error

$$\sup_x \left| \frac{1}{t}\Big(F(x,y+th, y'+th') - F(x,y,y')\Big) - \Big(F_y(x,y,y')h + F_{y'}(x,y,y')h'\Big) \right|$$

$$\leq \sup_x |U(t_0)| + |V(t_1)|$$

now, $x \mapsto F(x, y(x), y'(x))$ is continuous on the compact set $[x_i, x_{i+1}]$ where we have partition $[x_0, x_1] \cup \dots \cup [x_{n-1}, x_n]$ for $[a,b]$.

$\Rightarrow$ it is uniformly continuous, so

$$\sup_{\substack{x \\ t_0}} |U(x, t_0)| = \sup_{(-t,t)\ni t_0, x} |F_y(x, y+t_0h + y'+t_0h') - F_y(x,y,y')|$$

$$\leq \varepsilon \text{ for } t_0 \text{ sufficiently small,}$$

I.e in the estimate t is uniform in x.

- analogous for V

$\Rightarrow$ convergence $\frac{1}{t} F(x+yth, y'+th') - F(x,y,y') \xrightarrow[\text{is } C^0]{} F_y(x,y,y')h - F_{y'}(x,y,y')h'$

Now under uniform convergence we can exchange limits and the Riemann integral, hence

$$\lim_{t\to 0} \frac{(J(y+th) - J(y))}{t} = \int_a^b \lim_{t\to 0} \frac{1}{t}\Big(F(x, y+th, y'+th') - F(x,y,y')\Big)$$

$$= \int_a^b \Big(F_y(x,y,y')h + F_{y'}(x,y,y')h'\Big)\,dx$$

prop 1.2.2. if 1) and 2) from prop 1.1.2 hold, then it is clear $\delta J(y) : C_0^{1pw} \to \mathbb{R}$ is linear $\forall y \in D$

In particular, it is also Lipschitz (hence bounded):

$$\forall y \in D, \exists C(y) > 0, \quad \forall h \in C_0^{1pw} \quad |\delta J(y)h| < C(y)\|h\|_{1,pw}$$

— Note that if instead of

$$\forall y \in D, \forall h \in C_0^{1pw}, \quad \exists \varepsilon > 0 \quad \forall |t| < \varepsilon \quad y+th \in D$$

we take

$$\forall y \in D \quad \forall h \in C^{1pw} \quad \exists \varepsilon > 0 \quad \forall |t| < \varepsilon \quad y+th \in D$$

Then 1.2.1 & 1.2.2 are still valid.
Their proofs do not depend on the boundary conditions of h.

— Kielhöfer, exercise 1.2.2.

If in addition to $D \subset C^{1,pw}([a,b])$

$$\forall y \in D \ \forall h \in C^{1,pw}([a,b]) \ \exists \varepsilon > 0 \ \forall t : |t| < \varepsilon \quad y + th \in D$$

we have

F is continuous and two times continuously partially differentiable in y and y', then $\forall y \in D \ \forall h \in C^{1pw}$

$g''(0) = \delta^2 J(y)(h,h)$ exists and is

$$\delta^2 J(y)(h,h) = \int_a^b \left(F_{yy} h^2 + 2 F_{yy'} h h' + F_{y'y'} (h')^2 \right) dx$$

If in addition $F_{yy'}(\cdot, y, y') \in C^{1,pw}([a,b])$

then

$$\delta^2 J(y) = \int_a^b \left(P h^2 + Q (h')^2 \right) dx$$

where

$$P = F_{yy} - \frac{d}{dx} F_{yy'} \qquad Q = F_{y'y'}$$

$P, Q \in C^{0,pw}([a,b])$

more generally, if F_{yy}, $F_{y'y'}$, $F_{y'y}$ exist and are continuous, then

$$\lim_{t \to 0} \frac{g'_{h_1}(t) - g'_{h_1}(0)}{t} = \delta^2 J(y)(h_1, h_2)$$

$$= \int_a^b \left(F_{yy} h_1 h_1 + F_{yy'} (h_1 h_2' + h_2' h_1) + F_{y'y'} (h_1' h_2') \right) dx$$

is continuous, bilinear and bounded in terms of y:

$$| \delta^2 J(y)(h_1, h_2) | \leq C(y) \|h_1\|_1 \cdot \|h_2\|_1 \qquad \forall h_1, h_2 \in C^{1pw}$$

1.2.2 If also (in addition to the hypotheses of prop. 1.2.1) F is two times continuously partially differentiable wrt y and y', then: if you fix $y \in D$, $h \in C_0^{1,pw}[a,b]$,

with $\left.\begin{array}{l} g(t) := J(y+th) - J(y) \\ g : (-\varepsilon, \varepsilon) \to \mathbb{R} \end{array}\right\} \Rightarrow$ $g''(0)$ exists and equals
$$g''(0) = \int_a^b \left(F_{yy}(x,y,y')h^2 + F_{yy'}(x,y,y')hh' + F_{y'y'}(x,y,y')h'^2 \right) dx$$

proof By continuity of the 2nd order partial derivatives of $(y,y') \mapsto F(x,y,y')$ (for x fixed), we can apply Schwarz' lemma to conclude

$\forall y, y' \in \mathbb{R}$: $F_{yy'}(x,y,y') = F_{y'y}(x,y,y')$ for $\forall x$ fixed.

note that the functional $g' : (-\varepsilon, \varepsilon) \to \mathbb{R}$

$$g(t) := \int_a^b F_y(x, y(x)+th(x), y'(x)+th'(x))\, h(x) + F_{y'}(x, y(x)+th(x), y'(x)+th'(x))\, h'(x) \; dx$$

is known to exist and the integrand is uniformly continuous as a function of x, as it is continuous and defined on a compact set $[a,b] \subset \mathbb{R}$.

So we can take $\lim_{t \to 0} \frac{1}{t}(g'(t) - g'(0))$ and exchange the limit with the integral by uniform convergence, namely

$$\lim_{t\to 0} \frac{1}{t} \Big(F_y(x, y(x)+th(x), y'(x)+th'(x))h(x) + F_{y'}(x, y(x)+th(x), y'(x)+th'(x))h'(x) - \big(F_y(x,y(x),y'(x))h(x) + F_{y'}(x,y(x),y'(x))h'(x) \big) \Big)$$

$$= F_{yy}(x,y(x),y'(x))\,h(x)\,h(x) + F_{yy'}(x,y(x),y'(x))\,h'(x)h(x) + F_{y'y}(x,y(x),y'(x))\,h(x)\,h'(x) + F_{y'y'}(x,y(x),y'(x))\,h'(x)\,h'(x)$$

$$= F_{yy}(x,y,y')h^2 + F_{yy'}h'h + F_{y'y'}h'h'$$ uniformly

on a compact set $\Rightarrow \lim_{t\to 0} \frac{g'(t)-g'(0)}{t} = \int_a^b \dots \, dx$ $\boxtimes$

1.3.1 If $f \in C^{pw}[a,b]$, $\int_a^b f(x)h(x)\,dx = 0 \quad \forall h \in C_0^\infty[a,b)$

then $f \equiv 0$ on $[a,b]$

proof If not, then by $f \in C^{pw}[a,b]$ there is some open $I \subseteq [a,b]$ such that $f > 0$ on I.
(if $f < 0$, consider $-f$)

We can make a $C_0^\infty(a,b)$ - function h with $\left[\begin{array}{l} \text{supp}(h) \subset I, \\ \text{supp}(h) \neq \emptyset \end{array}\right]$
and then
$$\int_a^b f(x)h(x)\,dx = \int_I f(x)h(x)\,dx > 0.$$

How to find h?

$$g(x) = \begin{cases} e^{\frac{1}{x^2-1}} & x \in (-1,1) \\ 0 & x \notin (-1,1) \end{cases} \quad \text{has supp}(g) = [-1,1]$$

and is $C_0^\infty(\mathbb{R})$. Note it is also a classic example of a C^∞-, non-analytic function.

let $I = (c,d)$, then with $r = \frac{|c-d|}{2}$, $m = \frac{c+d}{2}$, $I = (m-r, m+r)$ and
$$h(x) := g\left(\frac{x-m}{r}\right)$$ is a function with supp$(h) = I$. Just restrict h to $[a,b]$.

$\square$

1.32 If $f \in C^{pw}[a,b]$ and $\int_a^b f(x)h'(x)\,dx = 0$

$\forall h \in C_0^{1,pw}[a,b]$, then $f \equiv c$ for some constant, piecewise on $[a,b]$

(so f can consist of different constant pieces)

proof. We need $\forall h \in C_0^{1,pw}[a,b]$ because we will construct a specific h from f to conclude this.

1.3 Fundamental lemma of calculus of variations

1.3.1 *lemma*

$f \in C^{0,pw}([a,b])$, and

$\forall h \in C_0^\infty((a,b)) \quad \int_a^b f h \, dx = 0$

then $f \equiv 0$ on $[a,b]$

def Here $C_0^\infty((a,b))$ are all ∞-times continuously diffable functions with compact support

$\mathrm{supp}(h) = \overline{\{h \neq 0\}} \subset (a,b)$

1.3.2 *lemma*

$f \in C^{0,pw}([a,b])$, and

$\forall h \in C_0^{1,pw}([a,b]) \quad \int_a^b f h' \, dx = 0$

then $f \equiv c$ on $[a,b]$ for some constant $c \in \mathbb{R}$

1.34 <u>Fundamental lemma of Calculus of Variations</u>

$f, g \in C^{0,pw}([a,b])$ and

$\forall h \in C_0^{1,pw}([a,b]) \quad \int_a^b (f h + g h') \, dx = 0$

Then 1) $g \in C^{1,pw}([a,b])$

2) $g' = f$ piecewise on $[a,b]$.

1.4 Euler - Lagrange Equations

def $y \in D \subset C^{1,pw}$ is a local minimizer of a functional $J: D \to \mathbb{R}$ if

$$\exists \delta>0, \ \forall \tilde{y}\in D: \|\tilde{y}-y\|_{1,pw} < \delta, \ J(\tilde{y}) \geq J(y)$$

if we require "—" $\forall \tilde{y}\in D$ $\|\tilde{y}-y\|_0 < \delta$, "——"

then the condition on δ has to hold for more $\tilde{y}$, so we call this y a strong local minimizer.

— Ex. 1.4.7: a local min. is not necessarily a strong local min.

namely, let $J(y) = \int_0^1 \left((y')^2 + (y')^3\right) dx$, $D = C_0^{1pw}[0,1]$

Then $y \equiv 0$ has $J(y) = 0$

And we can pick δ from the definition as follows: if $\|\tilde{y}\|_{1pw} < \delta$ then $\|\tilde{y}\|_0 + \|\tilde{y}\|_1 < \delta$

so $J(\tilde{y}) = \int_0^1 (y')^2 + (y')^3 \, dx$

now if $\|y'\|_1 < 1$ then $\tilde{y}' > -1$ on $[a,b]$ pw

$\Rightarrow (\tilde{y}')^2 + (\tilde{y}')^3 = (\tilde{y}')^2(1+\tilde{y}') > 0$

$\Rightarrow J(\tilde{y}) > 0$.

So y is a local minimizer @ $\delta = 1$.

However: take $y(x) = \begin{cases} \frac{h}{m}x & x\in[0,m] \\ h - \frac{h}{1-m}x & x \in [m,1] \end{cases}$

$\Rightarrow$ y is triangle with tip ~~centered~~ at m with height h.

it has $J(y) = \frac{h^2}{m} + \frac{h^3}{m^2} + \frac{h^2}{1-m} + \frac{h^3}{(1-m)^2}$

take m large, $m = 0.99$. Let $h \to \infty$
then $\mathcal{J}(y) \to -\infty$.

1.4.1 prop

If $y \in D \subset C^{1,pw}([a,b])$ is a local minimizer of
$$\mathcal{J}(y) = \int_a^b F(x, y(x), y'(x))\, dx$$

where F is cont. and cont. partially diffable wrt y and y'. Then

1) $F_{y'}(\cdot, y(\cdot), y'(\cdot)) \in C^{1,pw}([a,b])$

2) $\frac{d}{dx} F_{y'}(\cdot, y(\cdot), y'(\cdot)) = F_y(\cdot, y, y')$ piecewise on $[a,b]$

Note: if $y \in C^1([a,b])$, then

1) $F_{y'}(\cdot, y, y') \in C^1([a,b])$

2) $\frac{d}{dx} F_{y'} = F_y$ on $[a,b]$, not piecewise.

Proof: $\mathcal{J}$ satisfies the hypotheses of prop 1.2.2
hence $\delta \mathcal{J}(y)(h)$ exists $\forall h \in C_0^{1,pw}[a,b]$
and must be 0 since $g(t) = \mathcal{J}(y+th)$
attains a local min at $t=0$

$$\Rightarrow \int_a^b (F_y h + F_{y'} h')\, dx = 0 \quad \forall h \in C_0^{1,pw}([a,b])$$

$\Rightarrow$ fundamental lemma CoV:

$$\left\{ \begin{array}{l} F_{y'} \in C^{1,pw}([a,b]) \ \& \\ \frac{d}{dx} F_{y'} = F_y \end{array} \right.$$

however, we also have

1.42 prop

$$\left.\begin{array}{l} F_{y'} \in C^{1pw}(a,b], \\ \frac{d}{dx} F_{y'} = F_y \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \delta J(y)(h) = 0 \\ \forall h \in C_0^{1pw}([a,b]) \end{array}\right.$$

strong version of Euler-Lagrange

weak version of Euler-Lagrange.

Note: prop. 1.4.2 only holds in the case that y is a function of a 1-dim variable $x \in \mathbb{R}$.
It fails to hold for higher dim. spaces.

therefore, $\int_a^b fh + gh' \, dx = 0$ gives

$$\int_a^b (-F + g)\, h' \, dx = 0 \quad \forall h \in C_0^{1pw}[a,b]$$

$\Rightarrow$ by previous lemma, $g + -F \equiv c$ on $[a,b]$,
hence $g \equiv F + c \in C^{1,pw}[a,b]$.

So 1) $g \in C^{1pw}[a,b]$ and
2) $g' = (F+c)' = F' = f$ on $[a,b]$

$\square$

1.4.1 Euler lagrange:
if $y \in D$ is a local minimizer, then

$$\underbrace{\delta J(y)h = 0 \quad \forall h \in C_0^{1pw}[a,b]}_{\text{Strong version.}}$$

equivalent to $\int_a^b F_y(x,y,y')h + F_{y'}(x,y,y')h' \, dx = 0 \quad \forall h \in C_0^{1pw}[a,b]$

which by lemma 1.3.4 implies

weak version $\longrightarrow$ $\begin{cases} F_{y'} \in C^{1,pw}[a,b] \ \& \\ \frac{d}{dx} F_{y'} = F_y \text{ on } [a,b] \text{ (piecewise)} \end{cases}$

but in univariate case, weak v. $\Rightarrow$ strong v. due to:
if $\frac{d}{dx} F_{y'}$ exists piecewise on $[a,b]$
& $\frac{d}{dx} F_{y'} = F_y$, then $\forall h \in C_0^{1pw}[a,b]$,

$$\delta J(y)h = \int_a^b F_y(x,y,y')h + F_{y'}(x,y,y')h' \, dx =$$

$$\int_a^b F_y(x,y,y')h \, dx + \underbrace{F_{y'}(x,y,y')h \Big|_a^b}_{\substack{0 \\ \text{by } h \in C_0^{1pw}}} - \int_a^b \frac{d}{dx} F_{y'}(x,y,y')h \, dx$$

$$= \int_a^b \left(F_y - \frac{d}{dx}F_{y'}\right) h = \int_a^b 0\, h \, dx = 0 \quad \blacksquare$$

namely, define $h(x) := \int_a^x (f(\xi) - c)\, d\xi$

where $c = "\langle f \rangle" = \left(\int_a^b f(\xi)\, d\xi\right) / (b-a)$

why is $h \in C_0^{1,pw}$? it is $C^{1,pw}$ by the fundamental theorem of calculus, as $h' = f - c$

And $h(a) = 0$, $h(b) = \int_a^b f(\xi)\, d\xi - (b-a)c$

$= \int_a^b f(\xi) d\xi - \int_a^b f(\xi) d\xi$

$= 0$

Finally, by $h \in C_0^{1,pw}$ and the hypothesis,

$$\int_a^b (f - c) h' d\xi = \int_a^b f h'\, d\xi - \int_a^b c\, h'(\xi)\, d\xi$$

$$= 0 - c(h(b) - h(a))$$

$$= 0 - c(0 - 0) = 0$$

while also $\int_a^b (f-c) h'\, d\xi = \int_a^b (f(\xi) - c)^2 d\xi \geq 0$

with equality, only if $f(\xi) - c = 0$ <u>a.e.</u>

But $f - c \equiv 0$ <u>a.e.</u> and $f - c$ <u>pw. continuous</u> $\Rightarrow f \equiv c$ $\square$

1.3.4 If $f, g \in C^{pw}[a,b]$ and

$$\int_a^b f h + g h'\, dx = 0 \quad \forall h \in C_0^{1,pw}[a,b]$$

then

1) $g \in C^{1,pw}[a,b]$
2) $g' = f$ piecewise on $[a,b]$

proof by $f \in C^{pw}[a,b]$, it has a $C^{1,pw}$ primitive $F(x) = \int_a^x f(\xi)\, d\xi$ on $[a,b]$.

By partial integration, $\int_a^b f h\, dx = -\int_a^b F h'\, dx + F h \big|_b^a$

but by $h \in C_0^{1,pw}[a,b]$, $F h \big|_b^a = 0 - 0 = 0$

E.L. is necessary, not sufficient : Weierstraß' counterexample:

$$\mathcal{J}(y) = \int_a^b (y')^3 \, dx$$

Then EL gives $\frac{d}{dx} 3(y')^2 = 0$ and $3(y')^2 \in C^{1pw}$

$\Rightarrow$ $3(y')^2 = c$ on $[a,b]$

hence $y' = \pm\sqrt{c}$ piecewise on $[a,b]$

All sawtooth bridges between $(a,0)$, $(b,0)$ satisfy this.

However, there are none local minimizers :

$y' = \pm\sqrt{c}$ gives $\mathcal{J}(y) = (b-a)((t)\sqrt{c} + (1-t)\cdot -\sqrt{c})$

but $t = 1-t$ in order for y to start at $(a,0)$, end at $(b,0)$

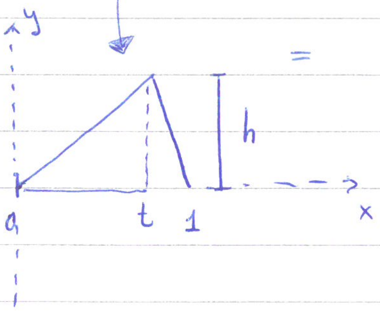
$\Rightarrow$ $\mathcal{J}(y) = 0$.

An example on $a=0$, $b=1$ (rescale appropriately) is

$$\tilde{y}(x) = \begin{cases} \frac{h}{t} x & x \in [0,t] \\ h - \frac{h}{1-t}(x-t) = h \cdot \frac{1-x}{1-t} & x \in [1-t, 1] \end{cases}$$

then
$$\mathcal{J}(y) = \left(\int_0^t \left(\frac{h}{t}\right)^3 dx + \int_t^1 \left(\frac{-h}{1-t}\right)^3 dx \right) = \frac{h^3}{t^2} - \frac{h^3}{(1-t)^2} = \frac{(1-t)^2 - t^2}{t^2(1-t)^2} h$$

$$= \frac{1-2t}{t^2(1-t)^2} \cdot h$$

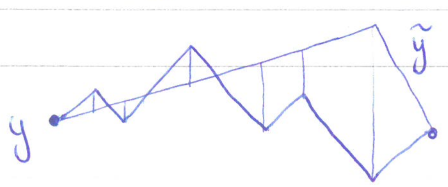


so if y sat. the EL equations,

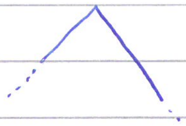
consider $y + \tilde{y}$, which has

$$(y+\tilde{y})' = \begin{cases} +\sqrt{c} + \frac{h}{t} \\ -\sqrt{c} + \frac{h}{t} \\ +\sqrt{c} - \frac{h}{1-t} \\ -\sqrt{c} - \frac{h}{1-t} \end{cases}$$

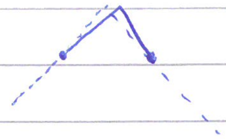
$\Rightarrow$ the integral depends on the overlap.



however, we can consider one peak



and nudge it to the right



so from $y \begin{cases} \frac{h}{2}x \\ \frac{h}{4} - \frac{1}{2}(x-\frac{1}{2}) \end{cases}$ we go to $\tilde{y} \begin{cases} \frac{h}{2t} \cdot x \\ \frac{h}{2}\left(\frac{1-x}{1-t}\right) \end{cases}$ $t > \frac{1}{2}$

then $|y-\tilde{y}|_1 = \ldots$ this approach is too direct

We can also take the approach: by calculus, $g(t) = J(y+th) - J(y)$, J having a min. at $y \Rightarrow g''(0) \geq 0$

and since the Lagrangian $F(x,y,y') = (y')^3$ is C^∞ in all its variables, $g''(0) = \delta^2 J(y)(h,h)$ indeed exists the $C_0^{1\,pw}[a,b]$, ~~hence~~ and it is

$$\int_a^b F_{yy} h^2 + 2F_{yy'} h'h + F_{y'y'} h'^2 \, dx =$$

$$\int_a^b (0 + 0 + 6y'h'^2)\, dx .$$

Now, see $y' = \pm\sqrt{c}$ piecewise on $[a,b]$, so if we let $h' = \begin{cases} 0 & \text{whenever } y' > 0 \\ \end{cases}$

whenever $y' < 0$ on an interval $[x_1, x_2]$, let $h = \begin{cases} c\cdot(x-x_1)/(t-x_1) \\ c\cdot\frac{x_2-x}{x_2-t} \end{cases}$

for some $x_1 < t < x_2$. Then

$$\int_{x_1}^{x_2} 6y'h'^2 = \int_{x_1}^{t} -6\sqrt{c}\left(c\frac{1}{t-x_1}\right)^2 + \int_t^{x_2} -6\sqrt{c}\left(c\frac{-1}{x_2-t}\right)^2 dx$$

$$= -6\sqrt{c}\cdot c^2\left(\frac{1}{t-x_1} + \frac{1}{x_2-t}\right) < 0 \quad \exists h\ \delta^2 J(y)(h,h) < 0$$

and $h \in C_0^{1\,pw}$ in this way. Therefore so not min

14.1 Prove for any compact $I \subseteq (a,b)$, $\exists$ sequence $(h_n)_{n\in\mathbb{N}} \subseteq C_0^{1,pw}[a,b]$ with

a) $\text{supp}(h_n) \subseteq I \quad \forall n$ b) $\lim_n \int_a^b h_n^2\,dx = 0$

c) $\lim_n \int_a^b (h_n')^2\,dx = \infty$.

proof : let $g(x) = \begin{cases} 0 & x \notin (-1,1) \\ e^{\frac{1}{2(x^2-1)}} & x \in (-1,1) \end{cases}$. Then $\text{supp}(g) = [-1,1]$ and $g(x) > 0$ only on $(-1,1)$ and $g \in C^\infty(\mathbb{R})$. For $I = (c,d)$ let $m = \frac{d+c}{2}$, $r = \frac{d-c}{2}$

then let $h_n(x) := g\left(\frac{x-m}{r/n}\right)$

- $\text{supp}(h_n) = \left(m - \frac{r}{n}, m + \frac{r}{n}\right) \subseteq I \quad \forall n$ (a)
- $\int_a^b h_n^2\,dx = \int_{m-\frac{r}{n}}^{m+\frac{r}{n}} h_n^2\,dx \leq \frac{2r}{n} \sup_{x\in[m-\frac{r}{n}, m+\frac{r}{n}]} |h_n^2(x)|$

$= \frac{2r}{n} h_n^2(m) \xrightarrow{=\frac{2r}{n}e} 0$ as $n \to \infty$ (b)

- $h_n'(x) = \frac{d}{dx} g\left(\frac{x-m}{r/n}\right) = \frac{n}{r} g'\left(\frac{x-m}{r/n}\right)$

$g'(x) = \frac{2x}{(2(x^2-1))^2} e^{\frac{1}{2(x^2-1)}}$

$\Rightarrow (g'(x))^2 = \frac{4x^2}{16(x^2-1)^4} e^{\frac{1}{2(x^2-1)}}$ .. complicated

We can also pick $h_n = \begin{cases} 0 & x \notin [c_n, d_n] \\ \frac{x-c_n}{t-c_n} & x \in [c_n, t] \\ \frac{d_n - x}{d_n - t} & x \in [t, d_n] \end{cases}$

for $t = \frac{c_n + d_n}{2}$

and $c_n = t + \frac{c-t}{n}$, $d_n = t + \frac{d-t}{n}$ $\Rightarrow t - c_n = \frac{d-c}{2n}$, $d_n - t = \frac{d-c}{2n}$

$\Rightarrow \text{supp}(h_n) = [c_n, d_n] \subset [c,d]$

$\lim_n \int_a^b h_n^2\,dx = 2\int_{c_n}^t \frac{x^2 - 2c_n + c_n^2}{(t-c_n)^2}\,dx$

$= \frac{2}{(t-c_n)^2}\left(\frac{x^3}{3} + (-2c_n + c_n^2)(t-c_n)\right)$

$=$

Special cases / conditions of / on the Lagrangian

1,5

7 F does not depend explicitly on x. And $y \in D \cap C^2(a,b)$ is a local minimizer

$\Rightarrow$ E.L. equation is satisfied

$$\frac{d}{dx} F_{y'}(y,y') = F_y(y,y') \qquad \text{on } [a,b]$$

due to $y \in C^2(a,b)$, we may compute

$$\begin{aligned}\frac{d}{dx}\left(F(y,y') - y' F_{y'}(y,y') \right) &= F_y(y,y')\, y' + F_{y'}(y,y')\, y'' - \left(y'' F_{y'}(y,y') + y' \frac{d}{dx} F_{y'}(y,y') \right) \\ &= y' \cdot \left(F_y(y,y') - \frac{d}{dx} F_{y'}(y,y') \right) \\ &= 0\end{aligned}$$

$\Rightarrow$ $\boxed{F - y' F_{y'} = c_1}$ for a $c_1 \in \mathbb{R}$ constant.

8 F does not depend explicitly on y, and $y \in D$ is a local minimizer.

Then $\frac{d}{dx} F_{y'}(\cdot, y') = 0$ piecewise on $[a,b]$

$\Rightarrow$ $F_{y'}(\cdot, y') = c_1$ on $[a,b]$.

9 F does not depend explicitly on y', and $y \in D$ is a local minimizer

Then $\frac{d}{dx} F_{y'} = F_y \Rightarrow F_y(x, y(x)) = 0$, which is not an ODE, but may locally have a unique sol. due to implicit fnc theorem.

1.51 Exercise.

Assume $F_{xx}\ F_{xy}\ F_{xy'}\ F_{yx}\ F_{y'x}$ $F_{yy}\ F_{yy'}\ F_{y'y}\ F_{y'y'}$ exist and are continuous

Let $y \in C^{1pw}[a,b] \cap C^1[x_1, x_2]$ be a solution of

EL eq. $\frac{d}{dx} F_{y'}(\cdot, y, y') = F_y(\cdot, y, y')$ on $[x_1, x_2]$

where $[x_1, x_2] \subseteq [a,b]$

If $F_{y'y'}(x_0, y(x_0), y'(x_0)) \neq 0$ for some $x_0 \in (x_1, x_2)$, prove that y is $C^2(x_0-\varepsilon, x_0+\varepsilon)$ for some $\varepsilon > 0$.

"Local ellipticity implies local regularity"

proof:

Since y, y' are continuous in $[x_1, x_2]$, $x_0 \in (x_1, x_2)$ and $F_{y'y'}$ is cont. in its 3 variables, therefore $x \mapsto F_{y'y'}(x, y(x), y'(x))$ cont. and nonzero in a neighbourhood of x_0, say $(x_0-\varepsilon, x_0+\varepsilon)$.

This means the function

$$\tilde{y}(x) = \Big(F_y(x, y(x), y'(x)) - F_{y'y}(x, y(x), y'(x))\, y'(x) - F_{y'x}(x, y(x), y'(x))\Big) \Big/ F_{y'y'}(x, y(x), y'(x))$$

is continuous in $(x_0-\varepsilon, x_0+\varepsilon)$.

but now notice that in $(x_0-\varepsilon, x_0+\varepsilon)$,

$$\underbrace{F_{y'y'}\,\tilde{y} + F_{y'y}\,y' + F_{y'x}} = F_y \overset{EL}{=} \underbrace{\frac{d}{dx} F_{y'}}$$

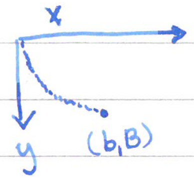
but if this is true, then $\tilde{y} = y''$ since it contradicts the chain rule otherwise

So y has a 2nd derivative on $(x_0-\varepsilon, x_0+\varepsilon)$ and it's $\tilde{y}$ and it is continuous.

$\boxtimes$

Brachistochrone problem

let $\tilde{A} = (0,0)$ $\tilde{B} = (b,B)$ be the endpoints, $B > 0$ $b > 0$ wlog.



minimize T, the time it takes for a frictionless point mass to travel along a curve $\mathcal{B}$ from $\tilde{A}$ to $\tilde{B}$, driven by gravity.

let $\mathcal{B}$ be parametrized by a C^1 curve $t \mapsto (x(t), y(t))$
$t \in [0,T]$

arc length $$s(t) = \int_0^t v(t) = \int_0^t \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt$$

$$\Rightarrow \frac{ds}{dt}(t) = v(t)$$

conservation of energy: $$\tfrac{1}{2} m v^2 + mg(h_0 - y) = mgh_0$$
$$\Rightarrow \tfrac{1}{2} m v^2 = mgy$$
$$\Rightarrow v(t) = \sqrt{2g\, y(t)}$$

assume that $\mathcal{B}$ also has a parametrization
$x \mapsto (x, \tilde{y}(x))$ $x \in [0,b]$

$$\Rightarrow^{\text{also}} s(t) = \int_0^{x(t)} \sqrt{1 + (\tilde{y}'(\xi))^2}\, d\xi$$

$$\Rightarrow \frac{ds}{dt}(t) = \frac{d}{dt} \int_0^{x(t)} \sqrt{1 + (\tilde{y}'(\xi))^2}\, d\xi$$

$$= \left(\frac{dx}{dt}(t)\right) \cdot \sqrt{1 + (\tilde{y}(x(t)))^2} - \left(\frac{d}{dt} 0\right) \cdot \sqrt{1 + \tilde{y}(0)^2}$$
$$+ \int_0^{x(t)} \left(\frac{\partial}{\partial t} \sqrt{1 + (\tilde{y}'(\xi))^2}\right) d\xi$$ Leibniz' rule

$$= \sqrt{1 + (\tilde{y}'(x(t)))^2} \cdot \dot{x}(t)$$

$\Rightarrow$ we have $v(t) = \sqrt{1 + (\tilde{y}'(x(t)))^2} \cdot \dot{x}(t)$
and $v(t) = \sqrt{2\, \tilde{y}(x(t))}$

$$\Rightarrow T = \int_0^T dt = \int_0^T \sqrt{\frac{1 + (\tilde{y}'(x(t)))^2}{2\, \tilde{y}(x(t))}}\, \dot{x}(t)\, dt$$

$$= \int_0^b \sqrt{\frac{1+(\tilde{y}'(x))^2}{2g\,\tilde{y}(x)}}\,dx$$

So we need to minimize

$$J(\tilde{y}) = \int_0^b \sqrt{\frac{1+(\tilde{y}')^2}{\tilde{y}}}\,dx$$

subject to the boundary conditions: $\tilde{D} =$

$$C[0,b] \cap C^{1,pw}(0,b] \cap \{\tilde{y}(0)=0 \quad \tilde{y}(b)=B\}$$
$$\cap \{\tilde{y}>0 \text{ on } (b,B]\} \cap \{J(\tilde{y})<\infty\}$$

we expect $\tilde{y}'(0)=\infty$

$J(\tilde{y})$ has a first variation on any interval bound away from 0:

for any $\delta>0$ $\exists\, d>0$ $\forall x\in[\delta,b]$ $\forall \tilde{y}\in\tilde{D}$ $\tilde{y}(x)>d$.
so for $h\in C_0^{1,pw}[\delta,b]$, with $\operatorname{supp}(h)\subset[\delta,b]$,
there is a $\varepsilon>0$ with $\forall t$ $|t|<\varepsilon$ $\tilde{y}+th\in\tilde{D}$
by continuity of the lagrangian on $[\delta,b]\times[d,\infty)\times\mathbb{R}$
and cont. diff-bility wrt $\tilde{y}$ and $\tilde{y}'$,
we have

$$\delta J(\tilde{y})h = \int_0^b F_{\tilde{y}}(\tilde{y},\tilde{y}')h + F_{\tilde{y}'}(\tilde{y},\tilde{y}')h'\,dx = 0$$

$\forall h\in C_0^{1,pw}[0,b]$

$\Rightarrow$ $\tilde{y}$ solves E.L. eq. piecewise on $[\delta,b]$

$$F_{\tilde{y}} = -\frac{1}{2\tilde{y}}\sqrt{\frac{1+(\tilde{y}')^2}{\tilde{y}}} \qquad F_{\tilde{y}'} = \frac{\tilde{y}'}{\sqrt{\tilde{y}(1+(\tilde{y}')^2)}}$$

by E.L, $f := F_{\tilde y'} \in C^{1,pw}[\delta,b]$, and

$$\frac{\tilde y'}{\sqrt{1+(\tilde y')^2}} = f\sqrt{\tilde y} \in C[\delta,b] \quad \text{, and } |f\sqrt{\tilde y}|<1$$

$$\Rightarrow \tilde y' = \sqrt{\frac{f^2\tilde y}{1-f^2\tilde y}} \in C[\delta,b] \quad \Rightarrow \tilde y \in C^{1,pw}$$

note $|f^2\tilde y|<1$

moreover, $F_{\tilde y'\tilde y'}(\tilde y,\tilde y') = \frac{1}{\sqrt{\tilde y}}\frac{1}{(1+(\tilde y')^2)^{3/2}} > 0$ on $(0,b]$

so exercise 1.5.1 implies $\tilde y \in C^2(0,b]$ for a minimizer.

By $\tilde y \in C^2(0,b]$, and 1.5. case 7, E.L. reduces to

$$F(\tilde y,\tilde y') - \tilde y' F_{\tilde y'}(\tilde y,\tilde y') = c_1 \quad , c_1 \in \mathbb{R}$$

$$\Rightarrow \sqrt{\frac{1+(\tilde y')^2}{\tilde y}} - \frac{(\tilde y')^2}{\sqrt{\tilde y(1+(y')^2)}} = c_1$$

$$\Rightarrow 1+(\tilde y')^2-(\tilde y')^2 = c_1\sqrt{\tilde y(1+(\tilde y')^2)}$$

$$\Rightarrow 1 = c_1^2(\tilde y(1+\tilde y'^2))$$

$$\Rightarrow \tilde y'^2 = \frac{1}{c_1^2\tilde y} - 1$$

$$\Rightarrow y' = \sqrt{\frac{1-c_1^2\tilde y}{c_1^2\tilde y}} = \sqrt{\frac{2r-\tilde y}{\tilde y}} \qquad 2r = \frac{1}{c_1^2}$$

There is no known special function solving this.

Bernoulli's approach: $(x,\tilde y(x)) = (\hat x(\tau), \hat y(\tau))$

where $\hat y(\tau) = r(1-\cos(\tau))$ [ansatz]

$$\Rightarrow \hat y(\tau) = \tilde y(\hat x(t)) \Rightarrow \hat y'(\tau) = \tilde y'(\hat x(t))\,\hat x'(\tau)$$

$$\Rightarrow \hat x'(\tau) = \frac{\hat y'(\tau)}{\tilde y'(\hat x(\tau))} = \frac{r\sin(\tau)}{\sqrt{\frac{2r-r+r\cos(\tau)}{r(1-\cos(\tau))}}}$$

$$= r\sin(\tau)\sqrt{\frac{1-\cos(\tau)}{1+\cos\tau}} = r(1-\cos(\tau))$$

use $\sin(\tau) = \sqrt{1-\cos^2\tau}$

$$\Rightarrow \quad \hat{x}(\tau) = \int_0^\tau r\,(1-\cos(t))\,dt = r(\tau - \sin(\tau)) + c_2$$
$$\hat{y}(\tau) = r\,(1-\cos(\tau)) \qquad \tau \in [\tau_0, \tau_b]$$

constanten uit randvoorwaarden:

$$\hat{y}(\tau_0) = r\,(1-\cos\tau_0) \stackrel{!}{=} 0 \quad \Rightarrow \quad \tau_0 = 0$$
$$\hat{x}(\tau_0) = \hat{x}(0) = c_2 \stackrel{!}{=} 0 \quad \Rightarrow \quad c_2 = 0$$
$$\left.\begin{array}{l} \hat{x}(\tau_b) = r\,\tau_b - \sin(\tau_b) = b \\ \hat{y}(\tau_b) = r\,(1-\cos(\tau_b)) = B \end{array}\right\}$$

hence $f(\tau_b) := \dfrac{\tau_b - \sin(\tau_b)}{1-\cos(\tau_b)} = \dfrac{b}{B}$

$f : [0, 2\pi)$ is strictly increasing with $f(0) = 0$
$f(t) \to \infty$ as $t \to 2\pi$. With $\frac{b}{B} > 0$, this implies
there is exactly one τ_b with $f(\tau_b) = \frac{b}{B}$.
and $r = B / (1 - \cos\tau_b)$

The resulting curve is a "cycloid"
$$\begin{cases} \hat{x}(\tau) = r\,(\tau - \sin(\tau)) & \tau \in [0, \tau_b] \\ \hat{y}(\tau) = r\,(1-\cos(\tau)) \end{cases}$$

Physical time?

$$t_0 = \int_0^{x(t_0)} \sqrt{\frac{1+(\tilde{y}')^2}{2g\tilde{y}}}\,d\xi = \int_0^{\tau_0} \sqrt{\frac{1+(\tilde{y}')^2}{2g\tilde{y}}}\,\hat{x}'(\tau)\,d\tau$$

$$= \left[\ \hat{y}(\tau) = \tilde{y}(\hat{x}(\tau)), \quad \tilde{y}'(x(\tau)) = \frac{\hat{y}'(\tau)}{\hat{x}'(\tau)} \right.$$

$$\int_0^{\tau_0} \sqrt{\frac{1 + \frac{\hat{y}'^2}{\hat{x}'^2}}{2g\hat{y}}}\,\hat{x}'\,d\tau$$

$$= \int_0^{\tau_0} \sqrt{\frac{1 + \left(\frac{1-\cos\tau}{\tau - \sin\tau}\right)^2}{2gr(1-\cos\tau)}}\, r(1-\cos(\tau))\,d\tau$$

$$= \int_0^{\tau_0} \sqrt{\frac{(\tau - \sin\tau)^2 + (1-\cos\tau)^2}{2gr(1-\cos\tau)}}\, r(1-\cos\tau)\,d\tau$$

$$\cancel{= \int_0^{\tau_0} \frac{r}{2g}\left(-2\tau\sin(\tau) - 2\cos(\tau) + \tau^2 + 2\right)(1-\cos(\tau))\,d\tau}$$

$$= \int_0^{\tau_0} \sqrt{\frac{\hat{x}'^2(\tau) + \hat{y}'^2(\tau)}{2g\hat{y}(\tau)}}\,d\tau \qquad \left[\begin{array}{ll} \hat{x}'(\tau) = & r(1-\cos\tau) \\ \hat{y}'(\tau) = & r\sin(\tau) \end{array}\right]$$

$$= \int_0^{\tau_0} \sqrt{\frac{r^2(1-2\cos(\tau)+\cos^2(\tau)+\sin^2(\tau))}{2rg(1-\cos(\tau))}}\,d\tau$$

$$= \sqrt{\frac{2r^2}{2rg}} \int_0^{\tau_0} \sqrt{\frac{1-\cos\tau}{1-\cos\tau}}\,d\tau = \sqrt{\frac{r}{g}}\,\tau_0$$

$\Rightarrow \quad t_0 = \sqrt{\frac{r}{g}}\,\tau_0$ in particular $T = \sqrt{\frac{r}{g}}\,\tau_b$

1.9 Natural Boundary conditions

let $J(y) = \int_a^b F(x, y(x), y'(x))\,dx$ be defined on all $D = C^{1,pw}[a,b]$, hence

$\forall y \in D \ \forall t \in \mathbb{R} \ \forall h \in C^{1,pw}[a,b] \qquad y + th \in D$

under additional assumption that F is cont. and F_y, $F_{y'}$ exist and are cont., we have

$\exists.\ \delta J(y)h = \int_a^b (F_y h + F_{y'} h')\,dx \qquad \forall h \in C^{1,pw}[a,b] \ \forall y \in D$

Prop. 1.9.1 Let y be a local minimizer of J. And let $D = C^{1pw}[a,b]$ without additional constraints.

Then in addition to E.L, y fullfills the "natural boundary conditions"

$$\begin{cases} F_{y'}(a, y(a), y'(a)) = 0 \\ F_{y'}(b, y(b), y'(b)) = 0 \end{cases}$$

proof strong version of E.L. gives:

$$0 = \delta J(y)h \overset{1.4.1}{=} \int_a^b (F_y h + F_{y'} h')\,dx$$

$$= \int_a^b \underbrace{\left(F_y - \frac{d}{dx}F_{y'}\right)}_{0 \text{ by E.L. weak version}} h\,dx + F_{y'} h \Big|_a^b$$

$$= F_{y'}(b, y(b), y'(b))\, h(b) - F_{y'}(a, y(a), y'(a))\, h(a)$$

$\Rightarrow$ with h s.t. $h(b)=0$ $h(a)\neq 0$
and the other way, we can show

$$F_{y'}(b, y(b), y'(b)) = 0 \qquad F_{y'}(a, y(a), y'(a)) = 0$$

! and if there is a boundary cond. such as
$y(a) = A$, then $h \in C^{1,pw} \cap \{h(a)=0\}$,
so we only have the result at b ("the other end")

$\square$

ex. let $J(y) = \int_a^b \sqrt{1+(y'(x))^2}\,dx$, the length of a curve between two lines $x=a$ and $x=b$.
Then Euler-Lagrange gives:

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0 \quad \text{piecewise on } [a,b]$$

$$\frac{\sqrt{1+y'^2}\,y'' - y'\frac{y'y''}{\sqrt{1+y'^2}}}{(1+y'^2)} = 0 \Rightarrow$$

$$(1+y'^2)y'' - y'^2 y'' = 0 \Rightarrow$$

$$y'' = 0 \Rightarrow$$

$$y(x) = c_1 x + c_2$$

The natural boundary conditions are

$$\frac{y'(a^b)}{\sqrt{1+(y'(a^b))^2}} = 0 \Rightarrow \left.\begin{array}{l} y'(a)=0 \\ y'(b)=0 \end{array}\right\} \Rightarrow c_1 = 0$$

so $y \equiv c_2$ a straight line.

For the Brachistochrone problem,
let's restrict to the one-sided boundary condition
$\tilde{y}(0) = 0$. We let $\hat{y}(b)$ free, hence
we get a natural boundary condition for $F_{\tilde{y}'}(b, \hat{y}(b), \hat{y}'(b))$

$$F = \sqrt{\frac{1+(\tilde{y}')^2}{\tilde{y}}} \quad \text{giving} \quad F_{y'}\big|_b = \frac{\tilde{y}'(b)}{\sqrt{\tilde{y}(b)(1+\tilde{y}'(b)^2)}} = 0$$

$$\Longrightarrow \tilde{y}'(b) = 0$$

$$\Rightarrow \hat{y}'(\tau_b) = \tilde{y}'(\hat{x}(\tau_b))\,\hat{x}'(\tau_b) = \tilde{y}'(b)\,\hat{x}'(\tau_b) = 0$$

$$\Rightarrow r\sin(\tau_b) = 0$$

therefore $\tau_b = n\pi$, $n = 1, \cancel{2}$ since $\mathcal{J}(2\pi) \uparrow$

$\Rightarrow \tau_b = \pi$ and we obtain cycloid

$$\begin{cases} \hat{x}(\tau) = \\ \hat{y}(\tau) \end{cases}$$

1.10 We can consider a larger class of curves γ in $\mathbb{R}^2$ by not restricting ourselves to graphs $y = y(x)$.

1.10.1 def A functional

$$\mathcal{J}(x,y) = \int_{t_a}^{t_b} \Phi(x,y,\dot{x},\dot{y})\,dt$$

defined on $(C^{1\,pw}[a,b])^2 \supset D$
with $\Phi : \mathbb{R}^4 \to \mathbb{R}$ continuous
is called a <u>functional in parametric form</u>

curves in D may have to sat. boundary cond. for $y(t_a)$, $x(t_a)$, ..

1.10.2 A parametric functional $\mathcal{J}$ is called <u>invariant</u> if $\Phi(x,y,\alpha\dot{x},\alpha\dot{y}) = \alpha\,\Phi(x,y,\dot{x},\dot{y})$

this means it is invariant under reparametrizations of t.

If $t \mapsto s(t)$ is a diffeomorphic bijection $[\tau_a,\tau_b] \to [t_a,t_b]$, then invariance means $(x\circ s)^{\cdot}(t) = \dot{x}(s(t))\dot{s}(t)$

$\Rightarrow$ $\cancel{\mathcal{J}(x,y) = \int_{t_a}^{t_b} \Phi(x,y,\dot{x},\dot{y})\,dt}$

$$\cancel{= \int_{t_a}^{t_b} \frac{\dot{s}(t)}{\dot{s}(t)} \Phi(x,y,\dot{x},\dot{y})\,dt}$$

$$\cancel{= \int_{t_a}^{t_b} \Phi(x(}$$

$$\int_{\tau_a}^{\tau_b} \Phi\left(x(s(t)), y(s(t)), \dot{\overline{x(s(t))}}, \dot{\overline{y(s(t))}}\right) dt$$

$$= \int_{\tau_a}^{\tau_b} \Phi\left(x(s(t)), y(s(t)), \dot{x}(s(t))\dot{s}(t), \dot{y}(s(t))\dot{s}(t)\right) dt$$

$$= \int_{\tau_a}^{\tau_b} \dot{s}(t)\, \Phi\left(x(s(t)), y(s(t)), \dot{x}(s(t)), \dot{y}(s(t))\right) dt$$

$$= \int_{t_a}^{t_b} \Phi\left(x(t), y(t), \dot{x}(t), \dot{y}(t)\right) dt$$

so $\mathcal{J}$ is "invariant under reparametrization"

ex. $\mathcal{J}(x,y) = \int_{t_a}^{t_b} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$. $\Rightarrow$ The length of a curve as defined by $\mathcal{J}$ does not depend on the parametrization $\gamma = (x,y) \in (C^{1pw}[a,b])^2$

ex. $\mathcal{L}(x,y) = K(x,y) - V(x,y)$

$= \frac{1}{2} m \sqrt{(\dot{x}^2 + \dot{y}^2)} - V(x,y)$

does depend on the parametrization $t \mapsto (x,y)(t)$.

prop 1.10.2 if Φ is continuously partially diff.able wrt $x, y, \dot{x}, \dot{y}$, then with $\mathcal{J}$ defined on $D \subseteq (C^{1pw}[a,b])^2$, we have:

$\forall (h_1, h_2) \in (C_0^{1pw}[a,b])^2$, the Gateaux diff exists and

$$d\mathcal{J}(x,y, h_1, h_2) = \int_{t_a}^{t_b} (\Phi_x h_1 + \Phi_y h_2 + \Phi_{\dot{x}} \dot{h}_1 + \Phi_{\dot{y}} \dot{h}_2) dx$$

it is clearly linear, so it is the first variation of $\mathcal{J}$ in (x,y) in direction (h_1, h_2), denoted $\delta \mathcal{J}(x,y)(h_1,h_2)$

prop 1.10.3 If a curve (x,y) minimizes $\mathcal{J}$, then $\delta \mathcal{J}(x,y)(h_1,h_2) = 0$ should clearly hold for any $(h_1, h_2) \in (C_0^{1pw}[a,b])^2$.

$\Rightarrow$ we get $\int_{t_a}^{t_b} \Phi_x h_1 + \Phi_y h_2 + \Phi_{\dot{x}} \dot{h}_1 + \Phi_{\dot{y}} \dot{h}_2 \, dt = 0$

choosing $h_1 \equiv 0$, $h_2 \equiv 0$ indep, we get the EL. equations

$\Rightarrow \int_{t_a}^{t_b} \Phi_x h_1 + \Phi_{\dot{x}} \dot{h}_1 \, dt = 0 \Rightarrow \frac{d}{dt} \Phi_{\dot{x}} = \Phi_x$ p.w.

in p. $\Phi_{\dot{x}} \in C^{1,pw}[a,b]$

and analogously $\frac{d}{dt} \Phi_{\dot{y}} = \Phi_y$

& $\Phi_{\dot{y}} \in C^{1,pw}[a,b]$. $\square$

By invariance under reparametrization, if Φ is invariant, we have, if $(x,y) \in (C^1[t_a,t_b])^2$, $(\tilde{x},\tilde{y}) \in (C^1[\tau_a,\tau_b])^2$

parametrize the same curve, that

$$\frac{d}{d\tau} \Phi_{\dot{x}}(\tilde{x},\tilde{y},\dot{\tilde{x}},\dot{\tilde{y}}) = \Phi_{\dot{x}}(x,y,\dot{x},\dot{y})\,\varphi(\tau)$$

$$\frac{d}{d\tau} \Phi_{\dot{y}}(\tilde{x},\tilde{y},\dot{\tilde{x}},\dot{\tilde{y}}) = \Phi_{\dot{y}}(x,y,\dot{x},\dot{y})\,\varphi(t)$$

hence (x,y) sat. EL $\iff$ $\tilde{x},\tilde{y}$ sat EL.

we can also generalize the natural boundary conditions: simply observe integr. by parts still holds:

prop 1.10.5

$$0 = \int_{t_a}^{t_b} \Phi_x h_1 + \Phi_y h_2 + \Phi_{\dot{x}} \dot{h}_1 + \Phi_{\dot{y}} \dot{h}_2 \, dt =$$

$$\int_{t_a}^{t_b} \left(\Phi_x - \frac{d}{dt}\Phi_{\dot{x}}\right) h_1 + \left(\Phi_y - \frac{d}{dt}\Phi_{\dot{y}}\right) h_2 \, dt$$

$$+ \Phi_{\dot{x}} h_1 \Big|_{t_a}^{t_b} + \Phi_{\dot{y}} h_2 \Big|_{t_a}^{t_b}$$

$$\Rightarrow \boxed{\Phi_{\dot{z}}}\Big|_{t_c} = 0 \quad \text{for } z \in \{x,y\},\ c \in \{a,b\}$$

(by choosing $h_1 \equiv 0$, $h_2 \equiv 0$ separately.)

conclusions depending on whether $x(t_a)$, $y(t_a)$.. need to sat. bdry conditions:

if $x(t_a) = A$, we must have $h(t_a) = 0$ therefore we cannot deduce $\Phi_{\dot{x}}(x(t_a), y(t_a), \dot{x}(t_a), \dot{y}(t_a)) = 0$ but the other 3 conclusions can still be derived.

$\square$

Digression: we can derive similar EL eq. and natural bdry cond. for $x \in (C^{1pw}[t_a,t_b])^n$. here $\Phi = \Phi(t,x,\dot{x})$

we call Φ invariant if $\Phi(x, \alpha\dot{x}) = \alpha\,\Phi(x,\dot{x})\ \forall x,\dot{x}$ and if Φ does not depend on t.

If $F: [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, then

$$\mathcal{J}(y) := \int_{t_a}^{t_b} F(t, x(t), \dot{x}(t))\, dt$$

defined $\mathcal{J}: C^{1,pw}[t_a, t_b] \to \mathbb{R}$, is continuous

proof fix $x \in C^{1pw}$ for $y \in C^{1,pw}$ we have

$$|\mathcal{J}(x) - \mathcal{J}(y)| = \left| \int_{t_a}^{t_b} F(t, x(t), \dot{x}(t)) - F(t, y(t), \dot{y}(t))\, dt \right|$$

$$\leq \int_{t_a}^{t_b} |F(t, x(t), \dot{x}(t)) - F(t, y(t), \dot{y}(t))|\, dt$$

$$\leq (t_b - t_a) \sup_{t \in [t_a, t_b]} |F(t, x(t), \dot{x}(t)) - F(t, y(t), \dot{y}(t))|$$

since F is cont. at x, for any $\varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ s.t: if $y \in C^{1,pw}$, $\|(t, x(t), \dot{x}(t)) - (t, y(t), \dot{y}(t))\| < \delta_\varepsilon$, then $|F(t, x(t), \dot{x}(t)) - F(t, y(t), \dot{y}(t))| < \varepsilon$.

So pick y such that $\|x - y\|_{1pw} = \sup_{t \in [t_a, t_b]} |x(t) - y(t)| + \sup_{t \in [t_a, t_b]} |x'(t) - y'(t)| < \delta_\varepsilon$.

then $\|(t, x(t), \dot{x}(t)) - (t, y(t), \dot{y}(t))\| \leq \|x - y\|_{1pw} < \delta_\varepsilon$ $\forall t \in [t_a, t_b]$

since this is $|x(t) - y(t)| + |\dot{x}(t) - \dot{y}(t)|$ ← sup

$\Rightarrow$ $|\mathcal{J}(x) - \mathcal{J}(y)| \leq (t_b - t_a)\, \varepsilon$ and this is arbitrarily small as $\varepsilon > 0$ arbitrarily small.

$\Rightarrow$ $\mathcal{J}$ is continuous at x.

Gateaux differential : existence proposition. 1.2.1

If $F : [t_a, t_b] \times \mathbb{R}^2 \to \mathbb{R}$ is cont. and cont partially differentiable wrt last two variables, and

$$J(y) := \int_{t_a}^{t_b} F(t, y(t), \dot{y}(t))\, dt$$

is defined on $D \subseteq C^{1,pw}[t_a, t_b]$ and
$\forall y \in D \quad \forall h \in C_0^{1,pw} \quad \exists \varepsilon > 0 \quad \forall t \in (-\varepsilon, \varepsilon) \quad y + th \in D$

Then $$dJ(y,h) = \int_{t_a}^{t_b} F_y(t, y(t), \dot{y}(t))\, h(t) + F_{\dot{y}}(t, y(t), \dot{y}(t))\, \dot{h}(t)\; dt$$

proof. denote $F = F(t, y(t), \dot{y}(t))$
$\tilde{F} = F(t, y(t) + sh(t), \dot{y}(t) + s\dot{h}(t))$

then $dJ(y,h) := \lim_{s \to 0} \frac{1}{s} \int_{t_a}^{t_b} \tilde{F} - F \; dt$ if it exists.

by F partially continuously diffable wrt y and $\dot{y}$,
$\tilde{F} - F$ is cont. diff.able wrt s and therefore
we can use Leibniz' integral rule to conclude existence

$$dJ(y,h) = \frac{d}{ds} \int_{t_a}^{t_b} (\tilde{F}) \, dt$$
$$= \int_{t_a}^{t_b} \partial_s \tilde{F} \, dt$$
$$= \int_{t_a}^{t_b} (F_y h + F_{\dot{y}} \dot{h}) \, dt .$$

$\square$