Complex Analysis, Summary

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0. Preliminaries

From Analysis 2, we use:

Theorem 0.1. ***uniform convergence preserves continuity***

*If $(f_n)_n$, with $f_n : X \to \mathbb{R}$ a function on a **compact metric space** X, then if $fN \to f$ uniformly, f is continuous.*

Definition 0.2. *For a sequence $(a_n) \subset V$ where V is a vector space with a metric $d : V \times V \to [0, \infty)$ (we need both a **metric** and **additive structure**), the **series** $\sum_{n=0}^{\infty} a_n$ **converges** if the sequence of partial sums $(\sum_{n=0}^{N} a_n)_N$ converges*

Corrolary 0.3. ***Weierstraß' M-test***

*If $(f_n)_n$, $f_n : X \to V$, $(M_n)_n \subset \mathbb{R}$ and X is **compact** and V is a **Banach space**, and $\forall n \in \mathbb{N},\ \forall x \in X : |f_n(x)| < M_n$, and $\sum_{n=0}^{\infty} M_n$ converges in $\mathbb{R}$, then $\sum_{n=0}^{\infty} f_n$ converges absolutely and uniformly on X.*

Proof. We first show the Cauchy criterion holds and use completeness of V to conclude. That is, we show:

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall m > n > N : \forall x \in X : \left| \sum_{k=n}^{m} |f_n(x)| \right| < \epsilon$$

This follows simply from the fact that we can bound $\sum_{k=n}^{m} |f_k(x)| < \sum_{k=n}^{m} M_k$ and $\sum_{n=0}^{\infty} M_n$ converges so we already have the Cauchy criterion for this series and can conclude.

Therefore, $\sum_{n=0}^{\infty} f_n$ converges absolutely pointwise, meaning to, say, $F : X \to V$. We next argue that this convergence is uniform: the sufficiently large N such that the Cauchy criterion is satisfied, can be picked from the sequence M_n, and thereby does not depend on x. □

Lemma 0.4. *Let $(V, |\cdot|)$ be a Banach (complete normed) space. If $\sum_{k=0}^{\infty} a_k$ converges absolutely, meaning that $\sum_{k=0}^{\infty} |a_k|$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.*

Proof. $S_N = \sum_{k=0}^{N} a_k$ converges if it is Cauchy, by completeness of V. We can show the Cauchy criterion: for $n \geq m$:

$$|S_n - S_m| = |\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k|$$

And since $\sum_{k=0}^{\infty} |a_k|$ converges, it is Cauchy, therefore we conclude $|S_n - S_m| < \epsilon$ for $n \geq m \geq N$ sufficiently large.

□

1 Complex Numbers

Definition 1.1. $\mathbb{C}$ ***defined as algebraic extension*** $\mathbb{C}/\mathbb{R}$

We observe that $f = X^2 + 1 \in \mathbb{R}[X]$ is a monic, irreducible polynomial, and $\mathbb{R}[X]$ is a principal ideal domain since $\mathbb{R}$ is a field. Therefore, there exists a field extension $\mathbb{R}/K$ with $\alpha \in K$ and $X^2 + 1$ the minimal polynomial of α, and in particular $K = \mathbb{R}[\alpha] \cong \mathbb{R}[X]/(X^2 + 1)$. It is unique up to isomorphism, and we denote it with $\mathbb{C}$, while we denote α with i. In other words, $\mathbb{C} = \mathbb{R}[i]$.

Definition 1.2. $\mathbb{C}$ ***as inner product space***

We can also see $\mathbb{C}^n$ as a vector space $\mathbb{C} \cong \mathbb{R}^{2n}$, which is an inner product space with $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ defined as:

$$\langle a + bi, c + di \rangle := ac + bd$$

Definition 1.3. $\mathbb{C}$ ***as normed space***

The inner product induces a norm $| \cdot | : \mathbb{C}^n \to [0, \infty)$ defined as:

$$|z| := \sqrt{\langle z, z \rangle}$$

*which is also called the **modulus** in the context of complex analysis*

Definition 1.4. $\mathbb{C}$ ***as metric space***

The norm induces a metric $d : \mathbb{C}^n \times \mathbb{C}^n \to [0, \infty)$

$$d(z, u) := |z - u|$$

Definition 1.5. ***conjugate***

*Note that if we define the **conjugate***

$$\overline{a + bi} := a - bi$$

then

$$\langle z, u \rangle = \sum_{k=0}^{n} z_k \overline{u_k} \implies \text{in } \mathbb{C},\ |z|^2 = z\overline{z}$$

and this immediately gives an expression for the inverse of any $z \in \mathbb{C}^*$, namely

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

Another view of $\mathbb{C}$ stems from the following field isomorphism:

$$\varphi : \mathbb{C} \to \mathrm{GL}_2(\mathbb{R})$$
$$\varphi(x+iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

Since φ satisfies the properties $\varphi(z+u) = \varphi(z)+\varphi(u)$ and $\varphi(zu) = \varphi(z)\varphi(u)$ and $f(1) = \mathrm{Id}_2$, we get that φ is a ring homomorphism. Since $\mathbb{C}$ is a field, $\ker\varphi$ is trivial and an isomorphism theorem gives:

$$\varphi(\mathbb{C}) \cong \mathbb{C}$$

And thereby

$$\varphi(\mathbb{C}) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$$

is a subfield of $\mathrm{GL}_2(\mathbb{R})$. Note that:

$$\det \circ \varphi = |\cdot|^2$$

Theorem 1.6. *The complex exponential function*

$$\exp : z \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is well-defined as the series converges, and is continuous.

Proof. We use Weierstraß' M-test ($\mathbb{C} \cong \mathbb{R}^2$ is a Banach space) to show the series converges on any compact disk of radius $R > 0$ around 0 in $\mathbb{C}$, that is $\overline{D}(0,R) := \{z \in \mathbb{C} \mid |z| \leq R\}$.

Indeed, setting $f_n : \overline{D}(0,R) \to \mathbb{C}$ through $f_n(z) := \frac{z^n}{n!}$, and choosing $M_n = \frac{R^n}{n!}$, we see $|f_n| \leq \frac{R^n}{n!}$ on D_R and $\sum_{n=0}^{\infty} \frac{R^n}{n!} = e^R$.

So $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely and uniformly on $\overline{D}(0,R)$ for any $R > 0$, therefore exp is well-defined. By uniform convergence, it follows we can exchange limits and the series, therefore it is continuous at any $z \in \mathbb{C}$, because for $a \in \overline{D}(0,R)$, we have $\lim_{z\to a} \exp(z) = \lim_{z\to a} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \lim_{z\to a} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n}{n!} = \exp(a)$. □

Theorem 1.7.

$$\exp(z+w) = \exp(z)\exp(w)$$

Proof.

$$\begin{aligned}\exp(z+w) &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \sum_{j,k\geq 0,\ j+k=n} \frac{1}{j!k!} z^k w^{n-k}\end{aligned}$$

Note that the above sum sums over every term $\frac{1}{k!j!} z^j w^k$ for each $(k,j) \in \mathbb{N} \times \mathbb{N}$ once. By absolute convergence we can apply Fubini's theorem, and it follows the series therefore equals

$$... = \sum_{(k,j)\in\mathbb{N}\times\mathbb{N}} \frac{z^k}{k!} \frac{w^j}{j!}$$

Next, we note that z and w both lie in a disk D_R of $R = \max\{|z|, |w|\}$, and $z + w$ mus certainly lie within the disk D_{2R}:

$$\begin{aligned}... &= \sum_{k\in\mathbb{N}} \frac{z^k}{k!} \sum_{j\in\mathbb{N}} \frac{w^j}{j!} \\ &= \exp(z)\exp(w)\end{aligned}$$

□

Exercise 1.8. *Prove that the following trigonometric and hyperbolic functions, now defined for complex variable by the series*

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \qquad \cosh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k)!}$$

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^{2k+1} z^{2k+1}}{(2k+1)!} \qquad \cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^{2k} z^{2k+1}}{(2k)!}$$

are well-defined and continuous in $z \in \mathbb{C}$.

Proof. The exp-function will do all the work, since we already know that this sequence converges. We can take termwise sums of convergent series, and this series necessarily also converges:

$$
\begin{aligned}
\frac{1}{2}\left(\exp(z)-\exp(-z)\right) &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{z^k}{k!}-\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\right)\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(1-(-1)^k)z^k}{k!}\\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{2z^{2k+1}}{(2k+1)!}\\
&= \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} = \sinh(z)
\end{aligned}
$$

In the same way, we can show that

$$
\begin{aligned}
\cosh(z) &= \frac{\exp z + \exp -z}{2}\\
\cos(z) &= \frac{\exp iz + \exp -iz}{2} = \cosh(iz)\\
\sin(z) &= \frac{\exp iz - \exp -iz}{2i} = -i\sinh(iz)
\end{aligned}
$$

□

Corrolary 1.9. ***Euler's identity***

$$\exp(iz) = \cos z + i\sin z$$

Definition 1.10. ***holomorphic functions***
*We say $f:\mathbb{C}\to\mathbb{C}$ is **complexly differentiable** at $z\in\mathbb{C}$ if there is a $C\in\mathbb{C}$ with*

$$\lim_{w\to z}\frac{f(w)-f(z)-C\cdot(w-z)}{w-z}=0$$

*We call C the **complex derivative** of f at z. A complex differentiable function is called **holomorphic**. Also, C is unique, namely*

$$C=\lim_{w\to z}\frac{f(w)-f(z)}{w-z}$$

And we define $f'(z)=C$.

Remark 1.11. *It is important to understand that having a complex derivative is a stronger property than just being differentiable on $\mathbb{C}$ seen as $\mathbb{R}^2$. Recall a function $f:\mathbb{R}^2\to\mathbb{R}^2$ is differentiable at $z\in\mathbb{R}^2$ if there is a 2×2-matrix D such that*

$$\lim_{w\to z}\frac{f(w)-f(z)-D\cdot(w-z)}{|w-z|}=0$$

While if we view a holomorphic function as a function $\mathbb{R}^2 \to \mathbb{R}^2$ (its real part being the first component and the imaginary part the second), then for the complex derivative $C = c_1 + c_2 i$ we have:

$$C \cdot (s_1 + is_2) = c_1 s_1 - c_2 s_2 + i(c_1 s_2 + c_2 s_1) \implies$$
$$\begin{pmatrix} c_1 s_1 - c_2 s_2 \\ c_2 s_1 + c_1 s_2 \end{pmatrix} = Ds = \begin{pmatrix} d_{11} s_1 + d_{12} s_2 \\ d_{21} s_1 + d_{22} s_2 \end{pmatrix} \implies$$
$$D = \begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix}$$

That is, f is holomorphic if and only

1. *if it is differentiable when seen as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$.*
2. *its differential $d_z f$ is a skew-symmetric matrix, namely given by $\varphi(\overline{C})$*

The skew-symmetry of the differential is a key property that makes holomorphic functions a special class of differentiable functions, and holomorphic functions are one of the key motivations of doing a study of the analysis of $\mathbb{C}$.

Definition 1.12. ***entirety***
*A function $f : \mathbb{C} \to \mathbb{C}$ is called **entire** if it is holomorphic at every $z \in \mathbb{C}$.*

Definition 1.13. ***regularity***
*A function $f : \mathbb{C} \to \mathbb{C}$ is called **regular** at $a \in \mathbb{C}$ if it is holomorphic at a and $f'(a) \neq 0$.*

Lemma 1.14. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and locally invertible at $z \in \mathbb{C}$, then f is regular at z.*

Proof. If f is locally invertible, i.e. there is an open $U \ni z$ and $g : f(U) \to U$ with $g(f(w)) = w$ for all $w \in U$, then we can use the inverse function theorem to argue that g is differentiable (at least through $\mathbb{R}$), and therefore we can apply the chain rule to obtain the equality $g'(f(z))f'(z) = 1$, which can only hold if $f'(z) \neq 0$. □

Theorem 1.15. *The function* exp *is entire and its derivative at any $z \in \mathbb{C}$ is* $\exp z$.

Proof. Note that for any $z \in \mathbb{C}$,

$$\frac{\exp(z+h) - \exp(z)}{h} = \exp(z)\frac{\exp(h) - 1}{h}$$

In the latter fraction, we use the series expansion and make the division termwise:

$$\frac{\sum_{n=0}^{\infty} \frac{h^n}{n!} - 1}{h} = \sum_{n=1}^{\infty} \frac{h^{n-1}}{n!}$$

Since the convergence of the series is uniform, we can take the limit $h \to 0$ through the summation. This means

$$\lim_{h \to 0} \sum_{n=1}^{\infty} \frac{h^{n-1}}{n!} = 1 + \sum_{n=2}^{\infty} \frac{0^{n-1}}{n!} = 1$$

(Where I made the constant term explicit to avoid confusing 0^0 discussions). We have, therefore, at every $z \in \mathbb{C}$:

$$\exp'(z) = \lim_{h \to 0} \exp(z) \cdot \frac{\exp(h) - 1}{h} = \exp(z)$$

□