Complex Analysis Homework Assignment 2

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Exercise 2.4 Let $f : \Omega \to \mathbb{C}$ be holomorphic on a connected domain $\Omega \subset \mathbb{C}$ and $\Delta = \partial^2_{xx} + \partial^2_{yy}$ be the Laplacian. Show

$$\Delta\left(|f(z)|^2\right) = 4|f'(z)|^2$$

Proof. Write $f = u + iv$ where $u = u(x, y)$, $v = v(x, y)$ where $x = \Re z$, $y = \Im z$. Since f is holomorphic, it is $C^2(\Omega)$ and therefore u and v are $C^2(\Omega)$, so applying Theorem 2.3, u and v are harmonic. We have:

$$\begin{aligned}
\Delta(|f|^2) &= \partial^2_{xx}u^2 + \partial^2_{xx}v^2 + \partial^2_{yy}u^2 + \partial^2_{yy}v^2 \\
&= 2u\partial^2_{xx}u + 2(\partial_x u)^2 \\
&\quad + 2v\partial^2_{xx}v + 2(\partial_x v)^2 \\
&\quad + 2u\partial^2_{yy}u + 2(\partial_y u)^2 \\
&\quad + 2v\partial^2_{yy}v + 2(\partial_y v)^2 \\
&= 2u\Delta u + 2v\Delta v + 2((\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2) \\
&= 2((\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2)
\end{aligned}$$

Where the final equality follows from Theorem 2.3 because u and v are harmonic, so their Laplacians vanish. Using Cauchy-Riemann's equalities: $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, we have:

$$\begin{aligned}
\Delta(|f|^2) = ... &= 2((\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2) \\
&= 2((\partial_x u)^2 + (-\partial_x v)^2 + (\partial_x v)^2 + (\partial_x u)^2) \\
&= 2(2(\partial_x v)^2 + 2(\partial_x u)^2) \\
&= 4((\partial_x v)^2 + (\partial_x u)^2)
\end{aligned}$$

While

$$4|f'|^2 = 4|\partial_x u + i\partial_x v|^2 = 4((\partial_x u)^2 + (\partial_x v)^2)$$

. This completes the proof: $\Delta(|f(z)|^2) = 4((\partial_x v(z))^2 + (\partial_x u(z))^2) = 4|f'(z)|^2$ □

Exercise 2.7 For a holomorphic map $f : \Omega \to \mathbb{C}$ show that the following are equivalent:

(i) f is constant;

(ii) $\Re f$ and $\Im f$ are constant;

(iii) $f' = 0$;

(iv) $|f|$ is constant;

Proof. (i) $\implies$ (ii): let f be constant, so there is a $z \in \mathbb{C}$ with $f(z) = c$ for all $z \in \Omega$. Then $\Re f(z) = \Re(c)$ for all $z \in \Omega$, therefore $\Re f$ is constant on Ω. Similarly $\Im f(z) = \Im(c)$ for all $z \in \Omega$, so $\Im f$ is constant on Ω. This proves (ii).

(ii) $\implies$ (iii). f is holomorphic, so it has a complex derivative, which is (due to Theorem 2.1) given by

$$f'(z) = \partial_x u(z) + i\partial_x v(z), \quad z \in \Omega$$

But u and v are $\Re f$ and $\Im f$ respectively, and these are assumed to be constant and therefore have $\partial_x u = 0$ and $\partial_x v = 0$. This implies $f' = \partial_x u + i\partial_x v = 0 + i0 = 0$ in the domain Ω.

(iii) $\implies$ (iv). $f' = 0$ in the domain. f can also be regarded as map $\Omega \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$, in which case it is differentiable (because it is holomorphic, and holomorphicity is stronger than differentiability in $\mathbb{R}^2 \to \mathbb{R}^2$) and its jacobian at all $z \in \Omega$ is

$$D_z f = \begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $f'(z) = c_1 + ic_2 = 0 + i0$. Now, $|f|$ may not be holomorphic as $z \mapsto |z|$ is not. However, $z \mapsto |f(z)|^2 = u^2(z) + v^2(z)$ where $f = u + iv$, is differentiable when regarded as a function $\Omega \to \mathbb{R}$, since u and v are differentiable. By $\partial_x u = \partial_x v = \partial_y u = \partial_y v = 0$ (see the Jacobian), we have:

$$\partial_x(|f|^2) = 2u\partial_x u + 2v\partial_x v = 0, \quad \partial_y(|f|^2) = 2u\partial_y u + 2v\partial_y v = 0$$

Therefore, $|f|^2$ has $\partial_x(|f|^2) = 0$ and $\partial_y(|f|^2) = 0$, and we know that it is totally differentiable, so from Analysis 2 it follows that $|f|^2$ is contant on Ω. From this it immediately follows that $|f| = \sqrt{|f|^2}$ is also constant on Ω.

(iv) $\implies$ (i): If $|f|$ is constant on Ω then $|f|^2$ is constant on Ω, therefore if $f = u + iv$ we have $|f|^2 = u^2 + v^2$ is constant. Since f is holomorphic, u and v have partial derivatives, and in particular since $\partial_x(|f|^2) = 0$ and $\partial_y(|f|^2) = 0$ (as f is a constant function $\Omega \to \mathbb{R}$, so use analysis 2 again), we get the equalities:

$$2u\partial_x u + 2v\partial_x v = \partial_x(|f|^2) = 0$$
$$2u\partial_y u + 2v\partial_y v = \partial_y(|f|^2) = 0$$

By holomorphicity, $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, and substituting this leads to

$$2u\partial_y v + 2v\partial_x v = \partial_x(|f|^2) = 0$$
$$-2u\partial_y v + 2v\partial_x v = \partial_x(|f|^2) = 0$$

Adding these equations gives $4v\partial_x v = 0$ on the domain. Since v and $\partial_x v$ are both continuous on Ω, this can only hold if $\{v = 0\} \cup \{\partial_x v = 0\} = \Omega$. Assume for a contradiction that there is an open $D \subset \Omega$ with $\partial_x v \neq 0$ on D. The openness is possible by continuity of $\partial_x v$. By $v\partial_x v = 0$, we need that $v = 0$ on D. This means v is constant on an open subset D of Ω. So $\partial_x v(z) = 0$ for an (interior) point $z \in D$ (Analysis 2, applied to a function $\Omega \to \mathbb{R}$), contradicting our assumption that $\partial_x v \neq 0$ on D. This shows $\partial_x v = 0$ everywhere. We had $\partial_x |f|^2 = 2u\partial_x u + 2v\partial_x v$, therefore $2u\partial_x u = 0$ because $\partial_x |f|^2 = 0$ and $2v\partial_x v = 0$. From $2u\partial_x u = 0$, it follows $\partial_x u = 0$ by the same argument as applied to v: assume $\partial_x u \neq 0$ somewhere, then it must be $\neq 0$ on an open set $D \subset \Omega$, therefore $u = 0$ on that open set, but then $\partial_x u(z) = 0$ for interior points $z \in D$, contradiction. Hence both $\partial_x u$ and $\partial_x v$ are 0 on Ω.

Finally $f' = \partial_x u + i\partial_x v = 0 + i0 = 0$ on Ω. This proves (ii), but we need to prove (i). (ii) $\implies$ (i) is precisely Theorem 2.4, so we conclude (i).

Having shown (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (i), we conclude equivalence. □

Exercise 3.2 Denote

$$M = \limsup_{n\to\infty} |a_n|^{1/n} \in [0, \infty]$$

Take $R = 1/M$ (so that $R = \infty$ when $M = 0$ and $R = 0$ when $M = \infty$). Prove that the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges for all $z \in \mathbb{C}$ if $R = \infty$; it converges for all $z \in D(a, R)$ if $R > 0$ is finite, and diverges for all $z \neq a$ if $R = 0$.

Proof. We use the root criterion for testing series convergence:

> *If* $(c_n)_{n=0}^{\infty}$ *is a sequence in* $\mathbb{R}$, *then the series* $\sum_{n=0}^{\infty} c_n$ *converges absolutely if* $\limsup_{n\to\infty} |c_n|^{1/n} < 1$, *diverges if* $\limsup_{n\to\infty} |c_n|^{1/n} > 1$.

- If $R = 0$, then $M = \infty$, in other words $(\sup_{k\geq n} |a_n|^{1/n})_{n\in\mathbb{N}}$ is unbounded. For $z = a$, we have $\sum_{n=0}^{\infty} a_n(z-a) = a_0$ converges, while for $z \neq a$, we have, for $c_n = a_n(z-a)^n$, that $|c_n|^{1/n} = |a_n|^{1/n}|z-a|$, and therefore

  $$\sup_{k\geq n} |c_k|^{1/k} = |z-a| \sup_{k\geq n} |a_k|^{1/k}$$

  Since $|z-a| > 0$, this sequence (in n) grows unboundedly, i.e. $\limsup_{n\to\infty} |c_n|^{1/n} = \infty$. Therefore, $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n(z-a)^n$ diverges and it follows the series diverges for $z \neq a$.

- If $R \in (0, \infty)$. Then if $|z-a| < R = 1/M$, i.e. $z \in D(a, R)$, we have (in the second equality, we can take the $|z-a|$ through lim sup because it is a nonnegative real number).

$$\begin{aligned}\limsup_{n\to\infty} |a_n(z-a)^n|^{1/n} &= \limsup_{n\to\infty} |z-a||a_n|^{1/n} \\ &= |z-a| \limsup_{n\to\infty} |a_n|^{1/n} < \frac{\limsup_{n\to\infty} |a_n|^{1/n}}{M} \\ &= \frac{M}{M} = 1\end{aligned}$$

hence by the root criterion, applied to the sequence $c_n = a_n(z-a)^n$, $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely for all $z \in D(a, R)$. If $|z-a| > R = 1/M$, then we see

$$\begin{aligned}\limsup_{n\to\infty} |a_n(z-a)^n|^{1/n} &= \limsup_{n\to\infty} |z-a||a_n|^{1/n} \\ &= |z-a| \limsup_{n\to\infty} |a_n|^{1/n} > \frac{\limsup_{n\to\infty} |a_n|^{1/n}}{M} \\ &= \frac{M}{M} = 1\end{aligned}$$

Therefore te series $\sum_{n=0}^{\infty} a_n(z-a)^n$ diverges by the root criterion. So we have absolute convergence for $z \in D(a, R)$ and divergence for $|z-a| > R$

- If $R = \infty$, then $M = 0$, hence if $z \in \mathbb{C}$ (which we could write as $|z-a| < \infty$), we have

$$\limsup_{n\to\infty} |a_n(z-a)^n|^{1/n} = \limsup_{n\to\infty} |0|^{1/n} = 0 < 1$$

hence by the root criterion, applied to the sequence $c_n = a_n(z-a)^n$, $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges absolutely for all $z \in \mathbb{C}$.

$\square$

Exercise 3.3

(a) For $z \in D(0, 1)$, compute the limit

$$\sum_{n=1}^{\infty} nz^n$$

(b) Show that for each $k > 0$, there exists a polynomial $P_k(t) \in \mathbb{Z}[t]$ with integer coefficients of degree $k-1$ such that

$$\sum_{n=0}^{\infty} n^k z^n = \frac{P_k(z)}{(1-z)^{k+1}}$$

Proof. (a) We already have that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ holds on the disk $D(0,1)$, and the latter is analytic at $a = 0$. Therefore, apply Theorem 3.2 to $f(z) = \frac{1}{1-z}$ and $R = 1$, $a = 0$ and the given series, so we can conclude that on the same disk:

$$\sum_{n=1}^{\infty} nz^n = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2}$$

(b) In (a), we showed the base case $k = 1$ for $P_1(z) = 1$ is a polynomial. Next, assume $k > 1$. By the induction hypothesis, we already have a $(k-2)$-degree polynomial P_{k-1} with integer coefficients such that:

$$\sum_{n=0}^{\infty} n^{k-1} z^n = \frac{P_{k-1}(z)}{(1-z)^k}$$

and that particular the right hand side converges on the disk $D(0,1)$. Theorem 3.2 gives that on $D(0,1)$:

$$\sum_{n=1}^{\infty} n^k z^{n-1} = \frac{d}{dz}\frac{P_{k-1}(z)}{(1-z)^k}$$

Note that we added the term nz^n for $n = 0$ since it is 0. Also, we can use convergence on the disk to take a factor $z \in D(0,1)$ through the series:

$$\sum_{n=0}^{\infty} n^k z^n = z\frac{d}{dz}\frac{P_{k-1}(z)}{(1-z)^k}$$

Finally, we obtain the derivative of the right side through the quotient rule, which is indeed valid for $z \in D(0,1)$ since $(1-z)^k \neq 0$ there:

$$\begin{aligned} z\frac{d}{dz}\frac{P_{k-1}(z)}{(1+z)^k} &= z\frac{-k(1-z)^{k-1}P_{k-1}(z) - (1-z)^k P'_{k-1}(z)}{(1-z)^{2k}} \\ &= \frac{-kzP_{k-1}(z) - z(1-z)P'_{k-1}(z)}{(1-z)^{k+1}} \end{aligned}$$

Clearly, $-kzP_{k-1}(z)$ has degree $k-2+1 = k-1$ and integer coefficients. $z(1-z)P'_{k-1}(z)$ too, since we take a derivative, and differentiation of a polynomial P yields a polynomial P' as derivative, and the coefficients stay integers as they are integer multiples of the coefficients of P, so differentiation is an operation $\mathbb{Z}[z] \to \mathbb{Z}[z]$ which decreases the degree of the polynomial by 1. We then add 2 to the degree by multiplication with $z(1-z)$ and this gives a degree $k-2-1+2 = k-1$ $\mathbb{Z}$-coefficient polynomial. We therefore get a recursion for P_k, namely

$$P_k(z) = -kzP_{k-1}(z) - z(1-z)P'_{k-1}(z)$$

The right hand side is an integer-coefficient polynomial of degree $k-1$ because $-kzP_{k-1}(z)$ and $z(1-z)P'_{k-1}(z)$ both are integer-coefficient polynomials of degree $k-1$

□

Exercise 4.3 Compute the integral of $f(z) = z^n$(on the domain $\mathbb{C}^\times$) over the unit disk for any (positive and negative) integer n.

Proof. For $n \geq 0$, $f(z) = z^n$ is holomorpic on a star-shaped domain, namely $\mathbb{C}$. Since the unit circle $S = \{x \in \mathbb{C} : |x| = 1\}$ is a closed contour, $\oint_S f(z)dz = 0$ by Theorem 4.2. This covers the case $n \geq 0$.

For $n < 0$, this does not hold since the function is not defined at $z = 0$. We perform the explicit calculation using the parametrization $\gamma : [0,1] \to S$, $\gamma(t) = e^{2\pi it}$:

For $n = -1$,

$$\begin{aligned}\oint_S f(z)dz &= \int_0^1 (e^{2\pi it})^{-1} d(e^{2\pi it}) \\ &= \int_0^1 2\pi i(e^{2\pi it})^{-1} e^{2\pi it} dt \\ &= \int_0^1 2\pi i dt = 2\pi i\end{aligned}$$

For $n < -1$, $n + 1 < 0$ and we get a different equality:

$$\begin{aligned}\oint_S f(z)dz &= \int_0^1 (e^{2\pi it})^n d(e^{2\pi it}) \\ &= \int_0^1 2\pi i(e^{2\pi it})^n e^{2\pi it} dt \\ &= \int_0^1 2\pi i e^{2(n+1)\pi it} dt \\ &= \left[\frac{2\pi i}{2\pi i(n+1)} e^{2\pi(n+1)it}\right]_0^1 \\ &= \frac{2\pi 1}{2\pi i(n+1)}(1-1) = 0\end{aligned}$$

□

Problem 1.11(a) Determine $a \in \mathbb{C}$ such that

$$u = e^{3x}\cos(ay)$$

is harmonic.

Proof.

$$\begin{aligned}\Delta u &= (\partial^2_{xx} + \partial^2_{yy})(e^{3x}\cos(ay)) \\ &= \cos(ay)\partial^2_{xx}e^{3x} + e^{3x}\partial^2_{yy}\cos(ay) \\ &= 9e^{3x}\cos(ay) - a^2e^{3x}\cos(ay) \\ &= (9 - a^2)e^{3x}\cos(ay)\end{aligned}$$

And we need $\Delta u = 0$, so this only holds if $a^2 = 9$, so necessarily $a = 3$ or $a = -3$. Note that these two choices give the same function u since $\cos(ay) = \cos(-ay)$. $\square$

Problem 1.11(c) Determine $a, b \in \mathbb{C}$ such that

$$u = ax^3 + by^3$$

is harmonic

Proof.

$$\begin{aligned}\Delta u &= (\partial_{xx}^2 + \partial_{yy}^2)(ax^3 + by^3) \\ &= \partial_{xx}^2 ax^3 + \partial_{yy}^2 by^3 \\ &= 6ax + 6by\end{aligned}$$

This is the zero function if and only if $a = -b$. There are no further restrictions on a and b, only that $a = -b$ and that $b \in \mathbb{C}$. $\square$