Complex Analysis Homework 1

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Exercise 1.1 Given a polynomial $p(z)$ with real coefficients, show that if z_0 is its zero then so is its conjugate $\overline{z_0}$. (In other words, complex roots always appear in conjugate pairs.) Conclude from this that a polynomial $p(z) \in \mathbb{R}[z]$ of odd degree always has a real root.

Proof. Let $p(z) = a_0 + a_1 z + ... + a_n z^n$ where $a_1, ..., a_n \in \mathbb{R}$. We are given $p(z_0) = 0$, so

$$a_0 + a_1 z_0 + ... + a_n z_0^n = 0$$

We can conjugate both sides:

$$\overline{a_0 + a_1 z_0 + ... + a_n z_0^n} = \overline{0} = 0$$

By the homomorphism property of conjugation, i.e. $\overline{ab + c} = \overline{a}\overline{b} + \overline{c}$, and the fact that $\overline{a_i} = a_i$ since every a_i is real, we get also that $\overline{a^n} = \overline{a}^n$ for any $n \in \mathbb{N}$. Using these identities, we can rewrite the above equation to:

$$\overline{a_0} + \overline{a_1 z_0} + ... + \overline{a_n z_0^n} = 0$$

$$\overline{a_0} + \overline{a_1}\ \overline{z_0} + ... + \overline{a_n}\ \overline{z_0}^n = 0$$

$$a_0 + a_1 \overline{z_0} + ... + a_n \overline{z_0}^n = 0$$

But the last one is just the equation $p(\overline{z_0}) = 0$, so indeed $\overline{z_0}$ is also a zero of p. $\square$

Exercise 1.2 Show that for all $z \in \mathbb{C}$,

$$\lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

Proof. We use the binomial theorem:

$$\begin{aligned}\left(1 + \frac{z}{n}\right)^n &= \sum_{k=0}^{n} \frac{1}{n^k} \binom{n}{k} \frac{z^k}{k!} \\ &= \sum_{k=0}^{n} \frac{(n)_k}{n^k} \frac{z^k}{k!}\end{aligned}$$

Here, $(n)_k$ is the "dalende faculteit" or Pochammer symbol $(n)_k := \frac{n!}{(n-k)!}$. So $(n)_k$ is a polynomial of degree k in n, with the leading coefficient equal to 1. This implies that $\frac{(n)_k}{n^k} \to 1$ as $n \to \infty$, for fixed k. The problem is that k is not fixed. However, we can solve this by noting that at least $\frac{(n)_k}{n^k}$ is bounded by 1, and $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ has the Cauchy criterion.

To be precise, pick for $\epsilon > 0$, a $N \in \mathbb{N}$ such that for all $m > N$ we have

$$\sum_{k=N}^{m} \frac{z^k}{k!} < \epsilon/2$$

Then we see, as $0 < (N)_k/N^k \leq 1$ for all $k, m \in \mathbb{N}$, that for all $m > N$:

$$\sum_{k=N}^{m} \frac{(m)_k}{m^k} \frac{z^k}{k!} < \epsilon/2$$

Next, we pick an $M \geq N$ with: for all fixed $k \leq N$ and all $m \geq M$,

$$\left| \frac{(m)_k}{m^k} \frac{z^k}{k!} - \frac{z^k}{k!} \right| < \epsilon/2N$$

This means: picking an $M \geq N$ with for all $m \geq M$

$$\left| \frac{(m)_k}{m^k} - 1 \right| < \frac{\epsilon k!}{2N|z|^k}, \ \forall k = 0, ..., N$$

which can be done indeed since $\frac{(n)_k}{n!} \to 1$ as $n \to \infty$, and we do this for only finitely many, namely N, k's, so we can pick the maximum M_k for which this holds. Then, we get the following approximation for any $n > M$:

$$\sum_{k=0}^{n} \frac{(n)_k}{n^k} \frac{z^k}{k!} = \sum_{k=0}^{N-1} \frac{(n)_k}{n^k} \frac{z^k}{k!} + \sum_{k=N}^{n} \frac{(n)_k}{n^k} \frac{z^k}{k!}$$

Therefore,

$$\begin{aligned}
\left| \sum_{k=0}^{n} \frac{(n)_k}{n^k} \frac{z^k}{k!} - \exp(z) \right| &= \left| \sum_{k=0}^{N-1} \frac{(n)_k}{n^k} \frac{z^k}{k!} + \sum_{k=N}^{n} \frac{(n)_k}{n^k} \frac{z^k}{k!} - \sum_{k=0}^{\infty} \frac{z^k}{k!} \right| \\
&\leq \sum_{k=0}^{N-1} \left| \frac{(n)_k}{n^k} \frac{z^k}{k!} - \frac{z^k}{k!} \right| + \left| \sum_{k=N}^{n} \frac{(n)_k}{n^k} \frac{z^k}{k!} \right| + \left| \sum_{k=N}^{\infty} \frac{z^k}{k!} \right| \\
&\leq \epsilon/2 + \left| \sum_{k=N}^{n} \frac{(n)_k}{n^k} \frac{z^k}{k!} \right| + \left| \sum_{k=N}^{\infty} \frac{z^k}{k!} \right| \\
&< \epsilon/2 + \epsilon/2 + \epsilon/2 = \frac{3}{2}\epsilon
\end{aligned}$$

And this is sufficient (we should have picked a bit smaller ϵ, but we can do this for $\frac{2}{3}\epsilon$ and conclude we can do it for any $\epsilon > 0$).

□

Exercise 1.4 Prove that the (multi-valued!) argument $\arg z$ satisfies

$$\arg(zw) = \arg z + \arg w \pmod{2\pi\mathbb{Z}}$$

Proof. Write $z = |z|e^{i\arg z}$ and $w = |w|e^{i\arg w}$. Since scaling by a positive real number doesn't change the argument of a complex number,

$$\arg(zw) = \arg(|z||w|e^{i\arg z}e^{i\arg w}) = \arg(e^{i\arg z}e^{i\arg w})$$

Next use the property $\exp(a+b) = \exp(a)\exp(b)$ for $a, b \in \mathbb{C}$, to get

$$\begin{aligned}\arg(e^{i\arg z}e^{i\arg w}) &= \arg(e^{i\arg z + i\arg w})\\ &= \arg(e^{i(\arg z + \arg w)})\end{aligned}$$

Finally, using $\arg(e^{i\theta}) = \theta \mod 2\pi$ for $\theta \in \mathbb{R}$:

$$= \arg z + \arg w + n \cdot 2\pi,\ n \in \mathbb{Z}$$

Any multiple of 2π can be used, as arg is defined modulo 2π and this gives the required equality:

$$\arg(zw) = \arg z + \arg w \pmod{2\pi}$$

□

Exercise 1.6 Show that the functions sin is entire and compute its derivatives.

Proof. We use the fact that linear combinations of functions holomorphic at $x \in C$ are holomorphic at x, since if

$$\lim_{y\to x}\frac{f(y)-f(x)}{|y-x|} \text{ and } \lim_{y\to x}\frac{g(y)-g(x)}{|y-x|}$$

both exist, then using the sum rule for limits, we derive

$$\lim_{y\to x}\frac{[f(y)+\lambda g(y)]-[f(x)+\lambda g(x)]}{|y-x|} = \lim_{y\to x}\frac{f(y)-f(x)}{|y-x|} + \lim_{y\to x}\frac{g(y)-g(x)}{|y-x|}$$

exists *and* is the sum of the complex derivatives. And if f g are entire, this means that they are holomorphic at every $z \in \mathbb{C}$ and therefore $f + \lambda g$ is holomorphic at every $z \in \mathbb{C}$, making $f + \lambda g$ entire.

Next, we note that exp is entire and that

$$\sin(z) = \frac{1}{2i}\exp(iz) - \frac{1}{2i}\exp(-iz)$$

Therefore, sin is entire as a linear combination of exps (note that if $\exp(iz)$ is entire as it is the composition of entire complex multiplication with i and exp). Moreover, its derivative is the sum of the derivatives of exp:

$$\frac{d}{dz}\sin(z) = \frac{1}{2i}\frac{d}{dz}\exp(iz) - \frac{1}{2i}\frac{d}{dz}\exp(-iz)$$

Now, using the chain rule, the individual terms are entire and their derivatives w.r.t. z are:

$$\begin{aligned}\frac{1}{2i}\frac{d}{dz}\exp(iz) - \frac{1}{2i}\frac{d}{dz}\exp(-iz) &= \frac{(i)}{2i}\exp(iz) - \frac{(-i)}{2i}\exp(-iz) \\ &= \frac{1}{2}\exp(iz) + \frac{1}{2}\exp(-iz) \\ &= \cos(z)\end{aligned}$$

Therefore, the derivative of sin at $z \in \mathbb{C}$ equals $\cos(z)$. The identities for cos and sin follow from Euler's identity:

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta) \implies \exp(i\theta) = \cos(\theta) - i\sin(\theta)$$

From which we indeed get $\cos(z) = \frac{e^{iz}+e^{iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ □

Problem 1.2(d) Solve and graph $z^4 + 16 = 0$.

Proof.

$$\begin{aligned}z^4 + 16 &= (z^2)^2 + 4^2 \\ &= (z^2)^2 - (4i)^2 \\ &\text{[merkwaardig product]} \\ &= (z^2 + 4i)(z^2 - 4i)\end{aligned}$$

Next, $x^2 = i$ has two solutions, namely

$$i = e^{\frac{\pi}{2}i}$$

and using this we get that both $\alpha_1 = e^{\frac{\pi}{4}i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $\alpha_2 = -e^{\frac{\pi}{4}i} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ have $\alpha_j^2 = i$, for $j = 1, 2$. From this, we derive that $i\alpha_1$ and $i\alpha_2$ have $(i\alpha_j)^2 = -i$, $j = 1, 2$. Finding roots of $z^2 \pm 4i = 0$ is now straightforward as we can just scale with a factor $\sqrt{4} = 2$:

$$z^2 - 4i = 0 \text{ has roots } z = \pm 2(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = \pm(\sqrt{2} + i\sqrt{2})$$

$$z^2 + 4i = 0 \text{ has roots } z = \pm 2i(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) = \mp(\sqrt{2} - i\sqrt{2})$$

This gives 4 solutions:

$$\sqrt{2} + i\sqrt{2}, \quad -\sqrt{2} - i\sqrt{2}, \quad \sqrt{2} - i\sqrt{2}, \quad -\sqrt{2} + i\sqrt{2}$$

Giving a factorization:

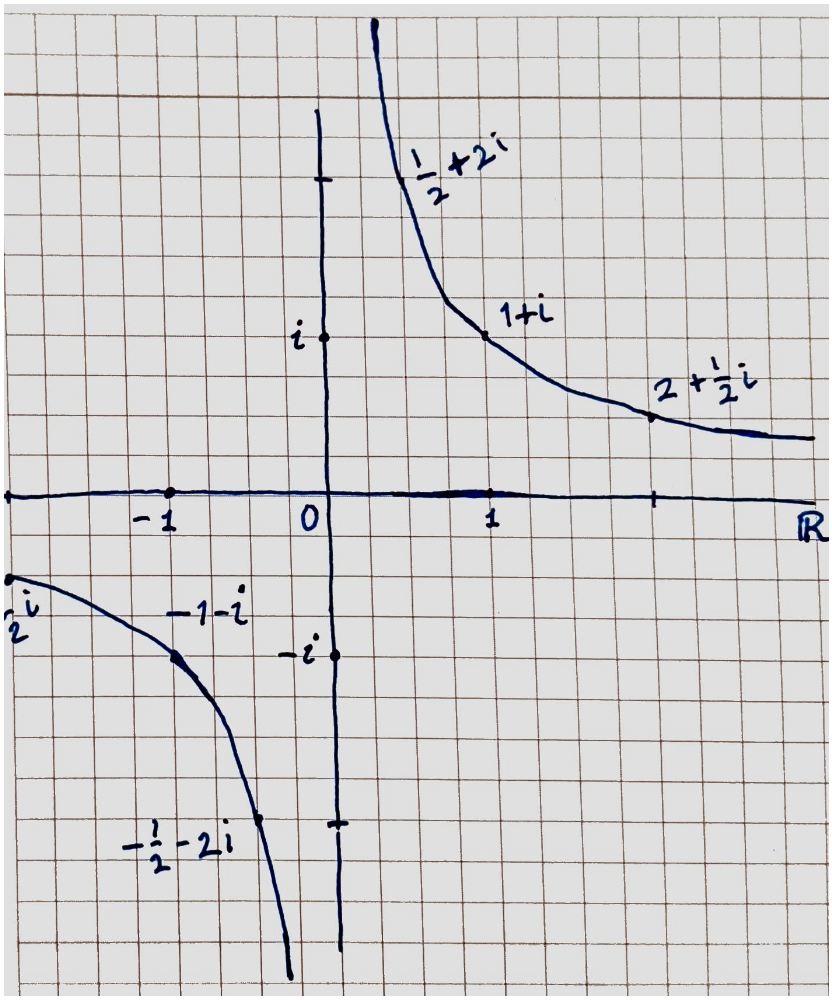
$$\begin{aligned}
z^4 + 16 &= (z - (\sqrt{2} + i\sqrt{2}))(z - (-\sqrt{2} - i\sqrt{2}))(z - (\sqrt{2} - i\sqrt{2}))(z - (-\sqrt{2} + i\sqrt{2})) \\
&= (z - \sqrt{2} - i\sqrt{2})(z + \sqrt{2} + i\sqrt{2})(z - \sqrt{2} + i\sqrt{2})(z + \sqrt{2} - i\sqrt{2}) \\
&= (z - \sqrt{2} - i\sqrt{2})(z - \sqrt{2} + i\sqrt{2})(z + \sqrt{2} + i\sqrt{2})(z + \sqrt{2} + i\sqrt{2}) \\
&\text{[merkwaardig product]} \\
&= ((z - \sqrt{2})^2 + 2)((z + \sqrt{2})^2 + 2) \\
&= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)
\end{aligned}$$

□

Problem 1.4(e)

$$\Im(z^2) = 2$$

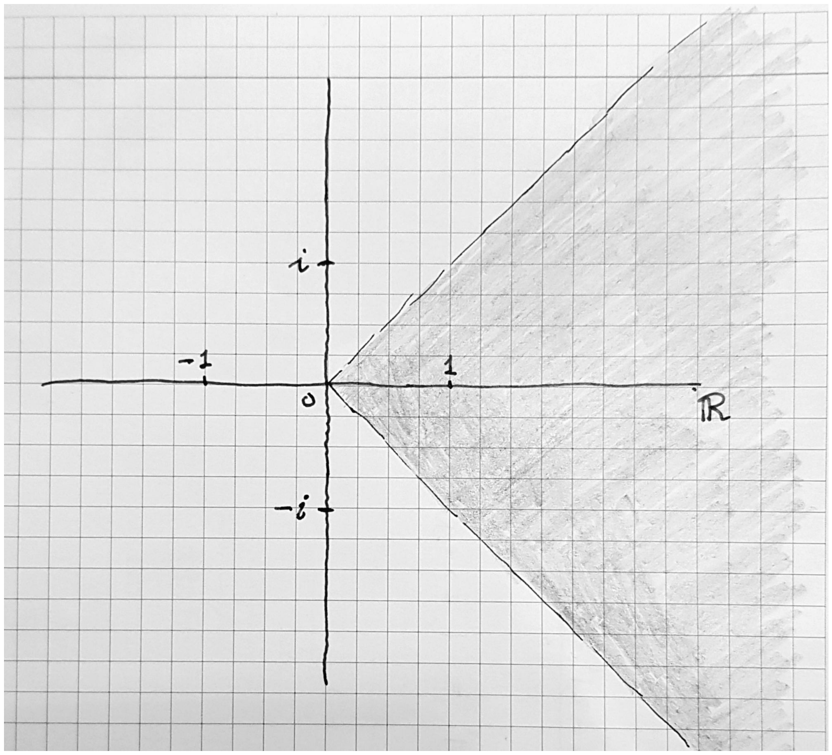
This holds for $z = a + bi$ $(a, b \in \mathbb{R})$ iff. $\Im((a + bi)(a + bi)) = 2ab = 2$, iff. $ab = 1$. For $(a, b) \in \mathbb{R}^2$, this is an hyperbola with tips $(1, 1)$ and $(-1, -1)$ and asymptotes the x-axis and y-axis, which are the real and the imaginary axis in $\mathbb{C}$:



Problem 1.4(h)

$$|\arg(z)| \leq \frac{\pi}{4}$$

This holds for $z \in \mathbb{C}$ if and only if $-\frac{\pi}{4} \leq \arg z \leq \frac{\pi}{4}$, which holds precisely for all imaginary numbers z that have $|\Im(z)| \leq \Re(z)$: these form a cone in the complex plane as follows:



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