Complex Analysis, Summary

Matthijs Muis

March 4, 2024

Acknowledgement My collection of chapter summaries is based on the **NWI-WB093: COMPLEX ANALYSIS (2023/24)** by Wadim Zudilin. I made almost no changes to the structure, except that I added a Preliminaries, and worked out the in-text exercises.

4 Contour Integrals and Cauchy Theorem

Definition 4.1. ***contour***

A ***contour*** *or* ***path*** *is a continuous mapping $\gamma : [a, b] \to \mathbb{C}$ from a compact interval $[a, b] \subset \mathbb{R}$, such that γ is piecewise C^1, that is there are $a = t_0 < ... < t_n = b$ with $\gamma \in C^1([t_{k-1}, t_k])$*

This enables us to define the contour integral:

Definition 4.2. ***contour integral***

The ***contour integral*** *of a function $f : \mathbb{C} \supset \Omega \to \mathbb{C}$ that is continuous on Ω along a contour $\gamma : [a, b] \to \Omega$ is*

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\gamma'(t)dt$$

It is important to note that the parametrization does not matter so much: the only things that will affect the value of the integral are the orientation of γ and the image set $\gamma([a, b])$.

Lemma 4.3. *Verify that if γ and μ are two C^1 parametrizations $\gamma : [a, b] \to \Omega$, $\mu : [c, d] \to \Omega$ with $\gamma(a) = \mu(c)$, $\gamma(b) = \mu(d)$ and $\gamma([a, b]) = \mu([c, d])$, then*

$$\int_\gamma f(z)dz = \int_\mu f(z)dz$$

Proof. Using the fact that γ is $C^1([a, b])$, we can conclude it is a diffeomorphism everywhere where it has a nonzero derivative. Since $\gamma'(t) = 0$ on a connected subinterval $[p, q]$ of $[a, b]$ means that γ is constant on that subinterval, and

$\gamma(q) = \gamma(p)$ therefore, we can write

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_z^p f(\gamma(t))\gamma'(t)dt \\
&+ \int_p^q f(\gamma(t))\gamma'(t)dt \\
&+ \int_q^d f(\gamma(t))\gamma'(t)dt \\
&= \int_z^p f(\gamma(t))\gamma'(t)dt + \int_q^d f(\gamma(t))\gamma'(t)dt
\end{aligned}$$

We can cut away $[p, q]$ and conclude that we can always reduce γ and μ such that they have nonzero derivative on a non-open subset of $[a, b]$. Splitting this up into separate arguments for each interval where $\gamma' \neq 0$ and $\mu' \neq 0$, we can assume wlog that $\gamma' \neq 0$ and $\mu' \neq 0$ on $[a, b]$, $[c, d]$ respectively. So they are diffeomorphisms, and $\mu^{-1} : \gamma([a, b]) \to [c, d]$ is a diffeomorphism too. This gives a diffeomorphism $s = \mu^{-1} \circ \gamma : [a, b] \to [c, d]$. We conclude that we can apply the substitution rule $s = s(t)$ with this diffeomorphism: namely $\mu(s(t)) = \gamma(t)$ so $\gamma'(t) = \mu'(s(t))s'(t)$

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\
&= \int_a^b f(\mu(s(t)))\mu'(s(t))s'(t)dt \\
&= \int_c^d f(\mu(s))\mu'(s)ds \\
&= \int_\mu f(z)dz
\end{aligned}$$

□

Lemma 4.4. ***fundamental inequality***

Let $f : \mathbb{C} \supset \Omega \to \mathbb{C}$ be continuous and $\gamma : [a, b] \to \Omega$ a contour. Let

$$M = \sup_\gamma |f(z)| = \max t \in [a, b] f(\gamma(t))$$

This is indeed a maximum: $f \circ \gamma$ is a continuous function and $[a, b]$ is compact. Then we have

$$\left| \int_\gamma f(z)dz \right| \leq l(\gamma)M$$

Where the length $l(\gamma) := \int_a^b \gamma'(t)dt$

Proof. Consider the two continuous functions:

$$u \mapsto F(u) = \int_a^u f(\gamma(t))\gamma'(t)dt$$
$$u \mapsto G(u) = \int_a^u \gamma'(t)dt$$

These are continuous and differentiable by the fundamental theorem of calculus applied to the real and imaginary components separately. The generalized mean value theorem gives that there is a $c \in [a, b]$ such that

$$G'(c)(F(b) - F(a)) = F'(c)(G(b) - G(a))$$

Clearly, this reads:

$$\gamma'(c) \int_a^b f(\gamma(t))\gamma'(t)dt = f(\gamma(c))\gamma'(c) \int_a^b \gamma'(t)dt$$

Giving, if we assume w.l.o.g. that $\gamma' \neq 0$ on $[a, b]$:

$$\int_a^b f(\gamma(t))\gamma'(t)dt = f(\gamma(c))l(\gamma)$$

This mean value result gives us a simple upper bound by taking the supremum over c and the modulus on both sides. □

Definition 4.5. *For $\gamma : [a, b] \to \mathbb{C}$ a contour, define $-\gamma : [-b, -a] \to \mathbb{C}$ through $(-\gamma)(t) = \gamma(-t)$. This gives*

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$$

Note that, like with vector fields on vector spaces of higher dimension than 1, it is not directly the case that a primitive of f on $\mathbb{C}$ exists:

$$f(z) = \frac{1}{z} \implies \int_{|z|=1} f(z)dz = \int_0^1 \frac{d(e^{2\pi it})}{e^{2\pi it}} = \int_0^1 2\pi i e^{2\pi it} e^{-2\pi it} dt = 2\pi i$$

The results about the existence of primitives are very similar to the results described by the ***Poincaré lemma***. The version of Poincarés lemma that I have seen this far says

Lemma 4.6. *If $\omega \in \bigwedge(\Omega)$ for Ω an open and simply connected domain in $\mathbb{R}^N$, then there is a $F \in C^1(\Omega)$, that is $F :$ with $dF = \omega$, that is*

$$\sum_{i=1}^N (\partial_i F)dx_i = \sum_{i=1}^N \omega_i dx_i \iff \forall i = 1, ..., N : \partial_i F = \omega_i$$

*if and only if ω is **closed**, and the domain is simply connected (every continuous closed curve can be contracted to a single point). A **closed** 1-form has $\forall i = 1, ..., N : \omega_i \in C^1(\Omega)$ and*

$$\forall i, j \in \{1, ..., N\} : \partial_i \omega_j = \partial_j \omega_i$$

But in our situation, we are not looking for a primitive $F : \mathbb{C} \to \mathbb{R}$ but a primitive $F : \mathbb{C} \to \mathbb{C}$.

Remark 4.7. *Write $f(z) = f(x + uy) = u(x, y) + iv(x, y)$, then with $dz = dx + idy$, $f(z)dz = (udx - vdy) + i(udy + vdx)$, and fdz has exterior derivative*

$$d(fdz) = [(-\partial_y u + \partial_x v) + i(\partial_x u - \partial_y v)]dx \wedge dy$$

Therefore, fdz has a primitive if and only if Ω is open and the Cauchy-Riemann equations are satisfied.

However, the notes take the touristic approach via the theorem of Cauchy-Goursat. I will just state that theorem here because the proof is very visual and you should be on Wikipedia rather than reading these notes anyway.

Lemma 4.8. ***Cauchy-Goursat***

Let $f : \Omega \to C$ be a holomorphic function on a connected domain $\Omega \subset \mathbb{C}$. Take a closed triangle $T \subset \Omega$ and consider its boundary $\gamma = \partial T$ as a contour in Ω traversed in positive direction. Then

$$\int_\gamma f(z)dz = 0$$

Definition 4.9. *We call a set $\Omega \subset V$ where V is a vector space, star shaped if there is a $a \in \Omega$ with for all $v \in V$, if $v \in \Omega$ then also the ray $a + [0, 1](v - a) \subset \Omega$*

The following theorem is then slowly obtained from these preliminaries:

Theorem 4.10. *Suppose that $f : \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic on a star-shaped domain Ω with centre a. Then*

$$F(z) = \int_{[a,z]} f(t)dt$$

is a holomorphic primitive of f. Here, $[a, z] = \{a + (z - a)t : t \in [0, 1]\}$.

Note that it immediately extends to a simply connected domain, if the segment $[a, z] = \{a + (z - a)t : t \in [0, 1]\}$ is contained in the domain. Simply pick a star-shaped set around the segment.

Theorem 4.11. ***Caucy integral theorem***

Let $f : \Omega \to C$ be a holomorphic function on a star-shaped domain $\Omega \subset \mathbb{C}$. Then

$$\oint_\gamma f(z)dz = 0$$

For any closed contour γ in Ω.