Complex Analysis, Summary

Matthijs Muis

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3 Power Series and Analytic Functions

Definition 3.1. ***analytic functions***
*$f : \mathbb{C} \supset \Omega \to \mathbb{C}$ is **analytic** at $c \in \Omega$ if there is a sequence $(a_n)_n$ and a $R > 0$ such that on the disk $D(c, R)$, the following series converges to $f(z)$:*

$$\sum_{k=0}^{\infty} a_n \cdot (z-c)^k = f(z)$$

If this equation is valid at all $c \in \Omega$, we say f is analytic on Ω

Analytic functions on a metric space Ω are always smooth (i.e. $C^\infty(\Omega)$). The converse need not to hold: Consider $\Omega = \mathbb{R}$ and

$$f(x) := \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

We see

$$f^{(n)}(x) = \begin{cases} 0 & x < 0 \\ p_n(y)e^{-y} & y = \frac{1}{x},\ x > 0 \end{cases}$$

Where for each $n \in \mathbb{N}$, $p_n(y) \in \mathbb{R}[y]$. Since we know $\lim_{y\to\infty} \frac{p(y)}{e^y} = 0$ for any $\mathbb{R}$-polynomial, it follows that $f^{(n)}$ exists and is continuous on Ω. Yet f has all its derivatives equal to zero at $x = 0$, so its power series representation at 0 is $\sum_{k\in\mathbb{N}} 0$, which clearly does not converge to $f(z)$ for $z > 0$. So f is not analytic at 0. Indeed, if Ω is a metric space, then any $f : \Omega \to \mathbb{R}$ that is 0 on an open subset of Ω must have power series representation 0 there, and it follows that any analytic function on Ω that is zero on an open subset, is the zero function.

Lemma 3.2. *On $D(0, 1)$, the series $\sum_{k=0}^{\infty} z^k$ converges absolutely to $\frac{1}{1-z}$*

Proof. Partial sums:

$$\sum_{k=0}^{N} z^k = \frac{1 - z^{N+1}}{1 - z}$$

If $|z| < 1$ then $|z|^{N+1} \to 0$ (Analysis 1), therefore $z \to 0$ too. So we can take the limit $z \to \infty$ above to conclude absolute convergence. Since $\mathbb{C}$ is complete, this implies convergence on the disk. □

Lemma 3.3. ***Abel's lemma***

If a power series $\sum_{k=0}^{\infty} a_k z^k$ converges at $z_0 \in \mathbb{C}$, then it converges absolutely in all $z \in D(0, z_0)$

Proof. For $\sum_{k=0}^{\infty} a_k z^k$ converges, it follows $a_k z_0^k \to 0$ as $k \to 0$. This means that $|a_k||z|^k$ is bounded in $\mathbb{R}$, say by C. For any $z \in D(0, z_0)$, we have

$$\sum_{k=0}^{\infty} |a_k z^k| = \sum_{k=0}^{\infty} |a_k||z_0|^k \frac{|z|^k}{|z_0|^k} \leq \sum_{k=0}^{\infty} C \frac{|z|^k}{|z_0|^k} \leq \frac{C}{1 - \frac{|z|}{|z_0|}}$$

The series $\sum_{k=0}^{\infty} |a_k z^k|$ being bounded implies its absolute converges by its monotonicity. □

Theorem 3.4. *If the power series $\sum_{k=0}^{\infty} a_k (z-a)^k$ converges in a disk $D(a, R)$, then it converges absolutely and uniformly on any closed disk $\overline{D(a, r)}$ with $r < R$*

Proof. We can w.l.o.g. assume $a = 0$, else we can consider the power series as a function of the variable $z - a$. Then, apply Abel's lemma to conclude for any $r < R$ that there is absolute convergence for all $|z - a| < r + \epsilon < R$, implying absolute convergence of the series on the disk $|z - a| \leq r$. This implies that with $M_n = a_n r^n$, the series $\sum_{n=0}^{\infty} M_n$ converges. By $a_n (z - a)^n \leq M_n$ on this *compact* disk, we can use the M-test of Weierstraß to conclude that $\sum_{k=0}^{\infty} a_k (z - a)$ converges uniformly on the disk. It also converges absolutely by the same argument applied to $|a_k (a - z)| \in [a - r, a + r]$. □

Theorem 3.5. *Assume that a function $f : \Omega \to \mathbb{C}$ is analytic at $a \in \Omega$ and represented by the power series*

$$\sum_{n=0}^{\infty} a_n (z - a)^n$$

in the disk $D(a, R)$. Then

$$f'(z) = \sum_{n=0}^{\infty} a_n (z - a)^{n-1}$$

and this series converges in the same disk.

In the proof we appeal to the **root test**:

Lemma 3.6. $\sum_{k=0}^{\infty} b_k$ *, let* $C = \limsup_{k\to\infty} \sqrt[n]{|b_k|}$*, which may be* $+\infty$*.*

- *If* $C < 1$ *then* $\sum_{k=0}^{\infty} b_k$ *converges absolutely.*
- *If* $C > 1$ *then* $\sum_{k=0}^{\infty} b_k$ *diverges.*

As a consequence for a power series $\sum_{k=0}^{\infty} a_k(z-a)^k$*, it converges for* $|z-a| < \frac{1}{C}$*.*

Proof. W.l.o.g. $a = 0$. Since $\limsup_{n\to\infty} \sqrt[n]{n} = 1$, the series $\sum_{n=0}^{\infty} a_n z^n$ and $f'(z) = \sum_{n=0}^{\infty} a_n z^{n-1}$ both have radius of convergence

$$\frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} = R$$

So indeed $\sum_{n=0}^{\infty} a_n z^{n-1}$ converges on the same disk.

Next, pick any $) < r < R$ and $|z| < r$. Then pick an N such that

$$\sum_{n>N} n a_n z^n < \epsilon$$

The remainder uses the triangle inequality: By openness of the disk $D(0, r)$ around z, in particular for $h \in D(0, |z| - r)$ we have $w = z + h \in D(0, r)$, we have:

$$
\begin{aligned}
|\frac{f(w)-f(z)}{w-z} - \sum_{n=1}^{\infty} na_n z^{n-1}| &= |\sum_{k=0}^{\infty} \frac{a_n w^n - a_n z^n}{w-z} - \sum_{n=0}^{\infty} na_n z^{n-1}| \\
&= |\sum_{k=0}^{\infty} a_n \left(\frac{w^n - z^n}{w-z} - nz^{n-1} \right) | \\
&= |\sum_{k=0}^{N} a_n \left(\sum_{k=0}^{n-1} w^k z^{n-1-k} - nz^{n-1} \right) | \\
&\leq |\sum_{n=0}^{N} a_n \left(\sum_{k=0}^{n-1} w^k z^{n-1-k} - nz^{n-1} \right) | \\
&\quad + \sum_{n>N} a_n \left(\sum_{k=0}^{n-1} |w|^k |z|^{n-1-k} + n|z|^{n-1} \right) \\
&< |\sum_{n=0}^{N} a_n \left(\sum_{k=0}^{n-1} w^k z^{n-1-k} - nz^{n-1} \right) | \\
&\quad + \sum_{n>N} a_n \left(\sum_{k=0}^{n-1} r^k r^{n-1-k} + nr^{n-1} \right) \\
&= |\sum_{n=0}^{N} a_n \left(\sum_{k=0}^{n-1} w^k z^{n-1-k} - nz^{n-1} \right) | \\
&\quad + \sum_{n>N} 2na_n r^n \\
&< |\sum_{n=0}^{N} a_n \left(\sum_{k=0}^{n-1} w^k z^{n-1-k} - nz^{n-1} \right) | \\
&\quad + 2\epsilon
\end{aligned}
$$

We now use continuity of the polynomial $w \mapsto \sum_{n=0}^{N} a_n \sum_{k=0}^{n-1} w^k z^{n-1-k}$ at z to conclude that there is a $\delta > 0$ with for all $w \in D(z, \min\{|z| - r, \delta\})$,

$$
|\sum_{n=0}^{N} a_n \left(\sum_{k=0}^{n-1} w^k z^{n-1-k} - nz^{n-1} \right) | < \epsilon
$$

From which it follows

$$
|\frac{f(w)-f(z)}{w-z} - \sum_{n=1}^{\infty} na_n z^{n-1}| < 3\epsilon
$$

For w in a $\delta(\epsilon)$-disk around z, and we conclude. □

Exercise 3.7. *(a) For $z \in D(0,1)$, compute the limit*

$$\sum_{n=1}^{\infty} nz^n$$

(b) Show that for each $k > 0$, there exists a polynomial $P_k(t) \in \mathbb{Z}[t]$ with integer coefficients of degree $k-1$ such that

$$\sum_{n=0}^{\infty} n^k z^n = \frac{P_k(z)}{(1-z)^{k+1}}$$

Proof. (a) We already have that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ holds on the disk $D(0,1)$. Therefore, we can conclude that on the same disk:

$$\sum_{n=1}^{\infty} nz^n = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2}$$

(b) We showed the base case $k = 1$. Next, assume $k > 1$ and that we already have a $k-2$-degree polynomial P_{k-1} with integer coefficients such that:

$$\sum_{n=0}^{\infty} n^{k-1} z^n = \frac{P_{k-1}(z)}{(1-z)^k}$$

and that particular the right hand side converges on the disk $D(0,1)$. The lemma gives that

$$\sum_{n=1}^{\infty} n^k z^{n-1} = \frac{d}{dz}\frac{P_{k-1}(z)}{(1+z)^k}$$

Note that we can add the term $n = 0$ since it is 0. Also, we can use convergence on the disk to take a factor $z \in D(0,1)$ through the series:

$$\sum_{n=0}^{\infty} n^k z^n = z\frac{d}{dz}\frac{P_{k-1}(z)}{(1+z)^k}$$

Finally, we obtain the derivative of the right side through the quotient rule:

$$\begin{aligned} z\frac{d}{dz}\frac{P_{k-1}(z)}{(1+z)^k} &= z\frac{-k(1-z)^{k-1}P_{k-1}(z) - (1-z)^k P'_{k-1}(z)}{(1-z)^{2k}} \\ &= \frac{-kzP_{k-1}(z) - z(1-z)P'_{k-1}(z)}{(1-z)^{k+1}} \end{aligned}$$

Clearly, $-kzP_{k-1}(z)$ has degree $k-2+1 = k-1$ and integer coefficients. So too has $z(1-z)P'_{k-1}(z)$ since we take a derivative, which also is an algebraic operation $\mathbb{Z}[z] \to \mathbb{Z}[z]$ which decreases the degree of the polynomial by 1. We even get a recursion for P:

$$P_k(z) = (kP_{k-1}(z) + P'_{k-1}(z))z + z^2 P'_{k-1}(z)$$

□