Complex Analysis, Summary

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2 Holomorphic Functions

We now turn the final remark of Chapter 1 into a theorem:

Theorem 2.1. ***Cauchy-Riemann equations***
If $f : \mathbb{C} \supset \Omega \to \mathbb{C}$, $f(z) = u(x, y) + iv(x, y)$ where $x = \Re(z)$, $y = \Im(z)$, is holomorphic at $c \in \mathbb{C}$, then:

$$\begin{aligned} \partial_x u(c) &= \partial_y v(c) \\ \partial_y u(c) &= -\partial_x v(c) \end{aligned}$$

And moreover:

$$\begin{aligned} f'(c) &= \partial_x u(c) + i\partial_x v(c) \\ &= \partial_y v(c) - i\partial_x u(c) \end{aligned}$$

On the other hand, if f satisfies the Cauchy-Riemann equations in $c \in \mathbb{C}$, then f is holomorphic at c.

Proof. Write $f(x + iy) = u(x, y) + iv(x, y)$, then if we regard f as a function $f : (x, y) \mapsto (u(x, y), v(x, y))$, i.e. a function $\mathbb{R}^2 \to \mathbb{R}^2$, then we can derive from

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} = C$$

two limits, one along the reals:

$$\begin{aligned} C &= \lim_{x \to z_1} \frac{u(x, y) + iv(x, y) - [u(z_1, z_2) + iv(z_1, z_2)]}{x + iz_2 - [z_1 + iz_2]} \\ &= \lim_{x \to z_1} \frac{u(x, y) + iv(x, y) - [u(z_1, z_2) + iv(z_1, z_2)]}{x - z_1} \\ &= \partial_x u(z_1, z_2) + i\partial_x v(z_1, z_2) \end{aligned}$$

And one along the imaginary axis:

$$
\begin{aligned}
C &= \lim_{y \to z_2} \frac{u(x,y) + iv(x,y) - [u(z_1,z_2) + iv(z_1,z_2)]}{(z_1 + iy - [z_1 + iz_2])} \\
&= \lim_{y \to z_2} \frac{u(x,y) + iv(x,y) - [u(z_1,z_2) + iv(z_1,z_2)]}{i(y - z_2)} \\
&= \frac{1}{i}\partial_y u(z_1,z_2) + \frac{1}{i} \cdot i\partial_y v(z_1,z_2) \\
&= -i\partial_y u(z_1,z_2) + \partial_y v(z_1,z_2)
\end{aligned}
$$

This now gives that the real and imaginary parts of these numbers should coincide and should equal C:

$$
\begin{aligned}
\partial_x u(z_1,z_2) &= \partial_y v(z_1,z_2) \\
-\partial_y u(z_1,z_2) &= \partial_x v(z_1,z_2)
\end{aligned}
$$

□

Exercise 2.2. *Prove the converse: if $f(x+iy) = u(x,y)+iv(x,y)$ is a function $f : \mathbb{C} \supset \Omega \to \mathbb{C}$, where u, v are $C^1(\{(x,y) : x + iy \in \Omega\})$ and satisfy the Cauchy-Riemann equations at $a \in \mathbb{R}^2$, then f is holomorphic at $a_1 + ia_2 \in \mathbb{C}$.*

Proof. Consider the Jacobian $J_a f$, which writes as

$$
J_a f = \begin{pmatrix} \partial_x u(a) & \partial_y u(a) \\ \partial_x v(a) & \partial_y v(a) \end{pmatrix} = \begin{pmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{pmatrix}
$$

Now consider $C = c_1 - ic_2$. Note the minus, it will reappear in the below fraction. Then, for any $z \in \mathbb{C}$, write $\delta = z - a$

$$
\begin{aligned}
\frac{f(z) - f(a) - C(z-a)}{z-a} &= \frac{u(z_1,z_2) + iv(z_1,z_2) - u(a_1,a_2) + iv(a_1,a_2) - (c_1\delta_1 + c_2\delta_2 + i(c_1\delta_2 - c_2\delta_1))}{z-a} \\
&= \frac{\begin{pmatrix} 1 & i \end{pmatrix} \left[\begin{pmatrix} u(z_1,z_2) \\ v(z_1,z_2) \end{pmatrix} - \begin{pmatrix} u(a_1,a_2) \\ v(a_1,a_2) \end{pmatrix} - \begin{pmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \right]}{z-a}
\end{aligned}
$$

And letting $z \to a$, this goes to:

$$
\begin{pmatrix} 1 & i \end{pmatrix} \lim_{z \to a} \frac{\begin{pmatrix} u(z) \\ v(z) \end{pmatrix} - \begin{pmatrix} u(a) \\ v(a) \end{pmatrix} - (J_a f) \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}}{z-a} = \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0
$$

Therefore, $C = c_1 - ic_2$ is indeed the complex derivative □

This means we don't need the complex numbers to define holomorphisms: they are simply functions $\mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ that are differentiable at $z \in \mathbb{R}^2$ such that $J_z f$ is a skew-symmetric matrix.

Exercise 2.3. *Check that the functions*

$$z \mapsto \Im z \qquad\qquad z \mapsto \Re z$$
$$z \mapsto |z| \qquad\qquad z \mapsto \overline{z}$$

are not holomorphic on $\mathbb{C}$*. At the same time, they are differentiable as functions from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$ *(when we exclude the origin for the last function).*

Proof. The problems already arise on the main diagonal:

- For $\Im$: $\partial_x u(z) = \partial_x 0 = 0$, while $\partial_y v(z) = \partial_y y = 1$.
- For $|\cdot|$: $\partial_x u(z) = \partial_x \sqrt{x^2+y^2} = \frac{x}{|z|}$, while $\partial_y v(z) = \frac{y}{|z|}$.
- For $\Re$: $\partial_x u(z) = x$, while $\partial_y v(z) = \partial_y 0 = 0$.
- For $\overline{z}$: $\partial_x u(z) = \partial_x x = 1$, while $\partial_y v(z) = \partial_y(-y) = -1$.

□

Corrolary 2.4. *If* $f, g : \mathbb{C} \supset \Omega \to \mathbb{C}$ *are holomorphic at* $z \in \mathbb{C}$ *. Then* $f+g$ *and* $f \cdot g$ *are holomorphic at* z *and*

$$(f+g)'(z) = f'(z) + g'(z)$$
$$(f \cdot g)'(z) = f'(z) \cdot g(z) + f(z) \cdot g'(z)$$

Proof. We immediately get by the ordinary sum- and chain rule that the sum $f+g$ or product $f \cdot g$ of two holomorphic functions f, g is differentiable with $D_z(f+g) = D_z f + D_z g$, $D_z(f \cdot g) = f(z) D_z g + g(z) D_z f$ Because matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

form a subring of $(GL_2(\mathbb{R}), +, \cdot, I_2, 0_2)$, we find that the Jacobians found from the original sum- and product rule are again skew-symmetric. We find the corresponding complex differentials $(f+g)'(z)$ and $(f \cdot g)'(z)$ by mapping these matrices to their corresponding complex number via the isomorphism

$$M \mapsto \overline{\varphi^{-1}(M)}$$

This means $(f+g)'(z) = f'(z) + g'(z)$ and $(f \cdot g)'(z) = f'(z) \cdot g(z) + f(z) \cdot g'(z)$. □

Definition 2.5. ***Conformal mappings***

We say $f; \mathbb{R}^d \supset \Omega \to \mathbb{R}^d$ *is* ***conformal*** *at* $a \in \Omega$ *if for any pair* γ_1, γ_2 *of smooth curves* $\gamma_1, \gamma_2 : I \to \mathbb{R}^d$ *(where* I *is an interval) passing through* a*, say* $\gamma_1(t_1) = \gamma_2(t_2) = a$*, the angle between the tangent vectors* $\gamma_1'(t_1)$*,* $\gamma_2'(t_2)$ *is preserved, in the sense that it equals the angle between* $(f \circ \gamma_1)'(t_1)$ *and* $(f \circ \gamma_2)'(t_2)$*:*

$$\frac{\langle \gamma_1'(t_1), \gamma_2'(t_2) \rangle}{|\gamma_1'(t_1)||\gamma_2'(t_2)|} = \frac{\langle (f \circ \gamma_1)'(t_1), (f \circ \gamma_2)'(t_2) \rangle}{|(f \circ \gamma_1)'(t_1)||(f \circ \gamma_2)'(t_2)|}$$

In $\mathbb{C}$, this writes as

$$\frac{\gamma_1'(t_1)\overline{\gamma_2'(t_2)}}{|\gamma_1'(t_1)||\gamma_2'(t_2)|} = \frac{(f\circ\gamma_1)'(t_1)\overline{(f\circ\gamma_2)'(t_2)}}{|(f\circ\gamma_1)'(t_1)||(f\circ\gamma_2)'(t_2)|}$$

Theorem 2.6. *If $f : \mathbb{C} \supset \Omega \to \mathbb{C}$ is regular and holomorphic at $a \in \mathbb{C}$, it is conformal at $a \in \mathbb{C}$.*

Proof. By Cauchy-Riemann, $J_a f$ is a skew-symmetric matrix $\begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix}$, which we can write as a constant times a rotation matrix, say $J_a f = rR = r \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Here $r > 0$ and in particular, we need $r \neq 0$, which follows from the assumption that f is regular. Clearly, Ru and Rv have the same angle between them as u and v (which can algebraically be verified!), and by the chain rule, $f \circ \gamma'(t) = (J_a f)(\gamma'(t)) = rR\gamma'$. So we get:

$$\begin{aligned}\frac{\langle (f\circ\gamma_1)'(t_1), (f\circ\gamma_2)'(t_2)\rangle}{|(f\circ\gamma_1)'(t_1)||(f\circ\gamma_2)'(t_2)|} &= \frac{\langle rR(\gamma_1'(t_1)), rR(\gamma_2'(t_2))\rangle}{r|R(\gamma_1'(t_1))|r|R(\gamma_2'(t_2))|} \\ &= \frac{r^2}{r^2}\frac{\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle}{|\gamma_1'(t_1)||\gamma_2'(t_2)|} \\ &= \frac{\langle \gamma_1'(t_1), \gamma_2'(t_2)\rangle}{|\gamma_1'(t_1)||\gamma_2'(t_2)|}\end{aligned}$$

Having seen what is so special about this situation in $\mathbb{R}^2$, Let's write this more shortly using notation in $\mathbb{C}$, where there is also the chain rule $(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$:

$$\begin{aligned}\frac{(f\circ\gamma_1)'(t_1)\overline{(f\circ\gamma_2)'(t_2)}}{|(f\circ\gamma_1)'(t_1)||(f\circ\gamma_2)'(t_2)|} &= \frac{f'(a)\gamma_1'(t_1)\overline{f'(a)\gamma_2'(t_2)}}{|f'(a)\gamma_1'(t_1)||f'(a)\gamma_2'(t_2)|} \\ &= \frac{\gamma_1'(t_1)\overline{\gamma_2'(t_2)}}{|\gamma_1'(t_1)||\gamma_2'(t_2)|}\end{aligned}$$

There seems to be much less going on. The proof in $\mathbb{R}^2$ was included to show the structure that is working in the background. □

Theorem 2.7. *Conversely, if $f : \Omega \to \mathbb{C}$ is conformal at $z \in \mathbb{C}$, then f is holomorphic at z.*

Proof. Since we assume that $f \circ \gamma : \mathbb{R} \to \mathbb{R}^2$ is differentiable in the $\mathbb{R}^2$-sense at t such that $\gamma(t) = z$ for any curve $\gamma \in C^1(\mathbb{R}, \mathbb{R}^2)$, we find that f is also differentiable, by approaching z along any curve γ. Then, we can apply the chain rule to get:

$$(f \circ \gamma)'(t) = (D_z f)\gamma'(t)$$

Now try the curves $\xi : t \mapsto (t, 0)$ and $\mu : t \mapsto (0, t)$, we get

$$\begin{pmatrix} \partial_x u(z) \\ \partial_x v(z) \end{pmatrix} = (f \circ \xi)'(t) = (D_z f)(\xi'(t)) = (D_z f) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} \partial_y u(z) \\ \partial_y v(z) \end{pmatrix} = (f \circ \mu)'(t) = (D_z f)(\mu'(t)) = (D_z f) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It follows that the Jacobian $D_z f$ must be

$$D_z f = \begin{pmatrix} \partial_x u(z) & \partial_y v(z) \\ \partial_x u(z) & \partial_y v(z) \end{pmatrix}$$

But this Jacobian has to be a conformal mapping $\mathbb{R}^2 \to \mathbb{R}^2$, and for $\mathbb{R}^2$ know that this is only the case if it is a rotation. From this follows the skew-symmetry of $D_z f$ and therefore f is holomorphic at z. □

Definition 2.8. ***harmonic function***
*A function $f : \mathbb{R}^d \supset \Omega \to \mathbb{R}^d$ is called **harmonic** if $f \in \mathbb{C}^2(\Omega)$ and:*

$$\Delta f = 0$$

Where $\Delta = \langle \nabla, \nabla \rangle = \sum_{k=1}^{d} \partial_{kk}^2$ (the Laplace operator).

Theorem 2.9. *If $f : \mathbb{C} \supset \Omega \to \mathbb{C}$ is holomorphic and $u = \Re f$, $v = \Im f$ are in $C^2(\Omega)$ then both u and v are harmonic functions on on $\Omega \subset \mathbb{R}^2$.*

Proof. By Cauchy-Riemann, $\partial_x u = \partial_y v$ and $\partial_y v = -\partial_x v$ on Ω. Together with *Schwarz'* theorem, which says that if $u \in C^(\Omega)$, then $\partial_k \partial_j u = \partial_j \partial_k u$ for any $j, k \in [d]$, it follows:

$$\partial_x \partial_x u = \partial_x(\partial_y v) = \partial_y \partial_x v = \partial_y(-\partial_y u) = -\partial_y \partial_y u \implies \Delta u = 0$$
$$\partial_x \partial_x v = \partial_x(-\partial_y u) = -\partial_x \partial_y u = -\partial_y \partial_x u = -\partial_y \partial_y v \implies \Delta v = 0$$

□

Theorem 2.10. *Vice versa, if u is harmonic on Ω, then there is a v, harmonic on Ω such that $f(z) = u(x, y) + iv(x, y)$ is holomorphic on Ω.*

Definition 2.11. *We call v the **harmonic conjugate** of u.*

Proof. We already have $u \in C^2(\Omega)$ and $\partial_x \partial_x u = -\partial_y \partial_y u$ on Ω. We want a $v \in C^2(\Omega)$ with $\partial_y v = \partial_x u$ on Ω. The approach will be an integral:

$$v(x, y) = \int_0^y \partial_x u(x, t) dt$$

Where we follow calculus conventions: take the integral over $[0, y]$ if $y > 0$, else take $-$ the integral over $[y, 0]$. This integral is well-defined since $\partial_x u(x, t)$

is continuous at $(x,t)\Omega$ and therefore in particular $t \mapsto \partial_x u(t,y)$ is continuous hence Riemann integrable.

By the fundamental theorem of calculus, we also get that:

$$\partial_y v(x,y) = -\partial_y u(x,y)$$

While by applying Leibniz' rule (which applies since $\partial_x u$ is continuously differentiable by $u \in C^2(\Omega)$):

$$\begin{aligned}\partial_x v(x,y) &= \partial_x \int_0^y \partial_x u(x,t)dt \\ &= \int_0^y \partial_x \partial_x u(x,t)dt\end{aligned}$$

Following the given $\partial^2_{xx} u = \partial^2_{yy} u$, and applying Leibniz's rule with respect to y (backward), then the fundamental theorem of calculus, we arrive at:

$$\begin{aligned}... &= \int_0^y -\partial_y \partial_y u(x,t)dt \\ &= -\int_0^y \partial_y \partial_y u(x,t)dt \\ &= -\partial_y \int_0^y \partial_y u(x,t)dt \\ &= -\partial_y u(x,y)\end{aligned}$$

And these are the Cauchy-Riemann equations, giving that $f = u+iv$ is holomorphic. Harmonicity of v follows from the fact that v is $C^2(\Omega)$ by the fundamental theorem of calculus. Therefore, we can directly apply the previous implication

$$f \text{ holomorphic}, \Im f, \Re f \in C^1(\Omega) \implies u, v \text{ harmonic}$$

□

Lemma 2.12. *Let $f : D \to \mathbb{C}$ be holomorphic on an open disk $D = D(a,r)$, centred at $a \in \mathbb{C}$ of radius $r > 0$. If $f'(z) = 0$ for all $z \in D$ then f is constant on D.*

Proof. Let $y \in D$, then let $\gamma : [0,1] \to \mathbb{C}$ be defined as the segment $\gamma(t) = a + t(y-a)$. This is a $\mathbb{C}^\infty$-curve, therefore there is a $t_0 \in [0,1]$ (by the mean value theorem) such that:

$$f(a) - f(y) = \frac{d}{dt} f(\gamma(t))|_{t=t_0} = f'(\gamma(t_0))\gamma'(t_0) = f'(a(1-t_0)+t_0 y)(y-a)$$

Now, the right hand size is 0 by $f'(z) = 0$ for all $z \in D$. So $f(y) = f(a)$ for all $y \in D$. □

Lemma 2.13. *Let $f : \Omega \to \mathbb{C}$ be holomorphic on an open connected set $\Omega \subset \mathbb{C}$. If $f'(z) = 0$ for all $z \in \Omega$ then f is constant on Ω.*

Proof. Let $y \in D$, then let $\gamma : [0,1] \to \mathbb{C}$ be a continuous arc $\gamma : [0,1] \to \mathbb{C}$ connecting a and y. Since Ω is open, it contains a an open disk $D(w,r)$ around every $w \in \gamma([0,1])$. Let $\{D_w\}_{w\in\gamma([0,1])}$ be such an open cover of γ. Since γ is continuous and $[0,1]$ is compact, $\gamma([0,1])$ is compact and we can extract a finite cover $D_{w_1} \cup ... \cup D_{w_n}$. This gives n segments lying in each D_{w_j}, connecting a with w_1, w_1 with w_2, ... , w_n with y. We can prove, using the preceding lemma, by induction that $f(w_j) = f(a)$ for every $j = 1, ..., n$, and next prove that $f(y) = f(w_n) = f(a)$. □

Corrolary 2.14. *If f and g are holomorphic maps on an open connected domain $\Omega \subset \mathbb{C}$ and $f' = g'$ on Ω, then $f - g$ is constant on Ω.*

Exercise 2.15. *For a holomorphic map $f : \Omega \to \mathbb{C}$ show that the following are equivalent:*

(i) *f is constant;*

(ii) *$\Re f$ and $\Im f$ are constant;*

(iii) *$f' = 0$;*

(iv) *$|f|$ is constant;*

Proof. (i) $\implies$ (ii): let f be constant, so there is a $z \in \mathbb{C}$ with $f(z) = c$ for all $z \in \Omega$. Then $\Re f(z) = \Re(c)$ for all $z \in \Omega$, therefore $\Re f$ is constant on Ω. Similarly $\Im f(z) = \Im(c)$ for all $z \in \Omega$, so $\Im f$ is constant on Ω. This proves (ii).

(ii) $\implies$ (iii). f is holomorphic, so it has a complex derivative, which is (due to Theorem 2.1) given by

$$f'(z) = \partial_x u(z) + i\partial_x v(z), \quad z \in \Omega$$

But u and v are $\Re f$ and $\Im f$ respectively, and these are assumed to be constant and therefore have $\partial_x u = 0$ and $\partial_x v = 0$. This implies $f' = \partial_x u + i\partial_x v = 0 + i0 = 0$ in the domain Ω.

(iii) $\implies$ (iv). $f' = 0$ in the domain. f can also be regarded as map $\Omega \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$, in which case it is differentiable (because it is holomorphic, and holomorphicity is stronger than differentiability in $\mathbb{R}^2 \to \mathbb{R}^2$) and its jacobian at all $z \in \Omega$ is

$$D_z f = \begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $f'(z) = c_1 + ic_2 = 0 + i0$. Now, $|f|$ may not be holomorphic as $z \mapsto |z|$ is not. However, $z \mapsto |f(z)|^2 = u^2(z) + v^2(z)$ where $f = u + iv$, is differentiable when regarded as a function $\Omega \to \mathbb{R}$, since u and v are differentiable. By $\partial_x u = \partial_x v = \partial_y u = \partial_y v = 0$ (see the Jacobian), we have:

$$\partial_x(|f|^2) = 2u\partial_x u + 2v\partial_x v = 0, \quad \partial_y(|f|^2) = 2u\partial_y u + 2v\partial_y v = 0$$

Therefore, $|f|^2$ has $\partial_x(|f|^2) = 0$ and $\partial_y(|f|^2) = 0$, and we know that it is totally differentiable, so from Analysis 2 it follows that $|f|^2$ is contant on Ω. From this it immediately follows that $|f| = \sqrt{|f|^2}$ is also constant on Ω.

(iv) $\implies$ (i): If $|f|$ is constant on Ω then $|f|^2$ is constant on Ω, therefore if $f = u + iv$ we have $|f|^2 = u^2 + v^2$ is constant. Since f is holomorphic, u and v have partial derivatives, and in particular since $\partial_x(|f|^2) = 0$ and $\partial_y(|f|^2) = 0$ (as f is a constant function $\Omega \to \mathbb{R}$, so use analysis 2 again), we get the equalities:

$$2u\partial_x u + 2v\partial_x v = \partial_x(|f|^2) = 0$$
$$2u\partial_y u + 2v\partial_y v = \partial_y(|f|^2) = 0$$

By holomorphicity, $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, and substituting this leads to

$$2u\partial_y v + 2v\partial_x v = \partial_x(|f|^2) = 0$$
$$-2u\partial_y v + 2v\partial_x v = \partial_x(|f|^2) = 0$$

Adding these equations gives $4v\partial_x v = 0$ on the domain. Since v and $\partial_x v$ are both continuous on Ω, this can only hold if $\{v = 0\} \cup \{\partial_x v = 0\} = \Omega$. Assume for a contradiction that there is an open $D \subset \Omega$ with $\partial_x v \neq 0$ on D. The openness is possible by continuity of $\partial_x v$. By $v\partial_x v = 0$, we need that $v = 0$ on D. This means v is constant on an open subset D of Ω. So $\partial_x v(z) = 0$ for an (interior) point $z \in D$ (Analysis 2, applied to a function $\Omega \to \mathbb{R}$), contradicting our assumption that $\partial_x v \neq 0$ on D. This shows $\partial_x v = 0$ everywhere. We had $\partial_x |f|^2 = 2u\partial_x u + 2v\partial_x v$, therefore $2u\partial_x u = 0$ because $\partial_x |f|^2 = 0$ and $2v\partial_x v = 0$. From $2u\partial_x u = 0$, it follows $\partial_x u = 0$ by the same argument as applied to v: assume $\partial_x u \neq 0$ somewhere, then it must be $\neq 0$ on an open set $D \subset \Omega$, therefore $u = 0$ on that open set, but then $\partial_x u(z) = 0$ for interior points $z \in D$, contradiction. Hence both $\partial_x u$ and $\partial_x v$ are 0 on Ω.

Finally $f' = \partial_x u + i\partial_x v = 0 + i0 = 0$ on Ω. This proves (ii), but we need to prove (i). (ii) $\implies$ (i) is precisely Theorem 2.4, so we conclude (i).

Having shown (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) $\implies$ (i), we conclude equivalence.

□

Exercise 2.16. *Let $f : \Omega \to \mathbb{C}$ be holomorphic on a connected domain $\Omega \subset \mathbb{C}$ and $\Delta = \partial^2_{xx} + \partial^2_{yy}$ be the Laplacian. Show*

$$\Delta\left(|f(z)|^2\right) = 4|f'(x)|^2$$

Proof. Write $f = u + iv$ where $u = u(x, y)$, $v = v(x, y)$ where $x = \Re z$, $y = \Im z$.

$$\begin{aligned}
\Delta(|f|^2) &= \partial^2_{xx}(u^2 + v^2) + \partial^2_{yy}(u^2 + v^2) \\
&= 2u\partial^2_{xx}u + 2(\partial_x u)^2 \\
&+ 2v\partial^2_{xx}v + 2(\partial_x v)^2 \\
&+ 2u\partial^2_{yy}u + 2(\partial_y u)^2 \\
&+ 2v\partial^2_{yy}v + 2(\partial_y v)^2 \\
&= 2u\Delta u + 2v\Delta v + 2((\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2) \\
&= 2((\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2)
\end{aligned}$$

While $4|f'|^2 = 4|\partial_x u + i\partial_x v|^2 = 4((\partial_x u)^2 + (\partial_x v)^2)$. We are almost there: just remove the ∂_y using Cauchy-Riemann's equalities: $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$ give:

$$\begin{aligned}
\Delta(|f|^2) = ... &= 2((\partial_x u)^2 + (\partial_y u)^2 + (\partial_x v)^2 + (\partial_y v)^2) \\
&= 2((\partial_x u)^2 + (-\partial_x v)^2 + (\partial_x v)^2 + (\partial_x u)^2) \\
&= 2(2(\partial_x v)^2 + 2(\partial_x u)^2) \\
&= 4((\partial_x v)^2 + (\partial_x u)^2)
\end{aligned}$$

□