Analysis 2, Chapter 8

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1 Inverse function theorem

Proposition 1. *Let $\Omega \subset \mathbb{R}^N$ be* ***open****, and $f : \Omega \to \mathbb{R}^M$* ***differentiable at*** *$x \in \Omega$. Then for any $S \subset \Omega$, it holds*

$$D_x f[T_x S] \subset T_{f(x)} f(S)$$

Proof. Let $v \in T_x S$, therefore we let $(x_n)_{n\in\mathbb{N}} \subset S$ be a sequence such that $x_n \to x$, $(\lambda_n)_{n\in\mathbb{N}} \subset (0, \infty)$ be a a sequence such that $\lambda_n \to 0$, and such that $\frac{x_n - x}{\lambda_n} \to v$.

If $x_n = x$ only for **finitely many** $n \in \mathbb{N}$, then **eventually** (for $n \geq N$, N sufficiently large) $x_n \neq x$, and we have:

$$\frac{f(x_n) - f(x) - (D_x f)(x_n - x)}{|x_n - x|} \to 0$$

By **differentiability**. From this, we can rewrite, for $n \geq N$:

$$\frac{f(x_n) - f(x)}{\lambda_n} = \frac{f(x_n) - f(x) - (D_x f)(v)}{|x_n - x|} \frac{|x_n - x|}{\lambda_n} + (D_x f)\left(\frac{x_n - x}{\lambda_n}\right)$$

Since $\frac{x_n - x}{\lambda_n} \to v$, we have $\frac{|x_n - x|}{\lambda_n} \to |v| < \infty$, and by **continuity** of the differential map $D_x f$, we have

$$\frac{f(x_n) - f(x) - (D_x f)(v)}{|x_n - x|} \frac{|x_n - x|}{\lambda_n} + (D_x f)\left(\frac{x_n - x}{\lambda_n}\right) \to 0 \cdot |v| + (D_x f)(v)$$

which implies convergence of $\frac{f(x_n) - f(x)}{\lambda_n}$ to $(D_x f)(v)$. If $x_n = x$ for infinitely many $n \in \mathbb{N}$, we have to conclude:

$$\forall n \in \mathbb{N} : \exists m \geq n : \frac{x_m - x}{\lambda_m} = 0 \text{ , therefore necessarily } v = \lim_{n\to\infty} \frac{x_n - x}{\lambda_n} = 0$$

On the other hand, for the same m, we have

$$\frac{f(x_m) - f(x)}{\lambda_m} = 0, \text{ , therefore necessarily } \lim_{n\to\infty} \frac{f(x_n) - f(x)}{\lambda_n} = 0 = (D_x f)(0)$$

So, in both cases, $\lim_{n\to\infty} \frac{f(x_n)-f(x)}{\lambda_n} = (D_x f)(v)$

Finally, notice that $(f(x_n))_{n\in\mathbb{N}}$ is a sequence **in** $f(S)$ **converging to** $f(x)$ by **continuity** of f. So by definition of the tangent cone, $(D_x f)(v)$ lies in $T_{f(x)}f(S)$. This proves the inclusion. $\square$

Definition 1. *A function $f : \mathbb{R}^N \to \mathbb{R}^M$ is called a **homeomorphism** on a set $S \subset \mathbb{R}^N$ if it is **continuous**, **invertible** and has a **continuous inverse.** This definition is used on general topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $f : X \to Y$, but on metric spaces in particular it means that*

$$x_n \to x \iff f(x_n) \to f(x)$$

For all $(x_n)_{n=1}^{\infty} \subset S$ and $x \in S$.

Remark 1. *The inclusion $D_x f[T_x S] \subset T_{f(x)}f(S)$ might be strict, and this is **not necessarily solved** when f is a homeomorphism: Let $f(x) = x^3 : \mathbb{R} \to \mathbb{R}$, then f is a homeomorphism with continuous inverse $x \mapsto \sqrt[3]{x}$. Yet $D_0 f = 0$, the zero map, hence $D_0 f[T_x S] = \{0\}$ for all $S \subset \mathbb{R}$. On the other hand, $T_{f(x)}f(\mathbb{R}) = \mathbb{R}$.*

*The inequality is strict because $D_0 f$ is **not injective**. We need that $D_x f$ is injective $\forall x \in \Omega$. This is proved and explored more carefully in the next section.*

1.1 Injective differential

Proposition 2. *If $\Omega \subset \mathbb{R}^N$ is **open** and $f : \Omega \to \mathbb{R}^M$ is **differentiable at** $x \in \Omega$, and $D_x f : \mathbb{R}^N \to \mathbb{R}^M$ is **injective on a linear subspace** $V \subset \mathbb{R}^N$, that is*

$$D_x f|_V : V \to \mathbb{R}^M$$

*is **injective**. Then, $f|x + V : x + V \to \mathbb{R}^M$ is **locally injective**: in particular, **there exists a** $\delta > 0$ s.t.*

$$\delta|x - y| \leq |f(x) - f(y)|, \quad \forall y \in B_\delta(x) \cap (x + V)$$

*Moreover, if f is C^1 on a neighbourhood of x, this holds **uniformly**:*

$$\delta|z - y| \leq |f(z) - f(y)|, \quad \forall y, z \in B_\delta(x) \cap (x + V)$$

Proof. Assume, for a contradiction, the negation of this, for $\delta = \frac{1}{n}$:

$$\forall n \in \mathbb{N} : \exists x_n \in B_{\frac{1}{n}}(x) \cap (x + V) : |x_n - x| > n \cdot |f(x_n) - f(x)|$$

Or,

$$\forall n \in \mathbb{N} : \exists x_n \in B_{\frac{1}{n}}(x) \cap (x + V) : \frac{|f(x_n) - f(x)|}{|x_n - x|} < \frac{1}{n}$$

Now, since $x_n \in B_{\frac{1}{n}}(x)$, $x_n \to x$. Moreover, for all $n \in \mathbb{N}$, $\frac{x_n - x}{|x_n - x|} \in \mathbb{S}^{N-1}$, which is a compact set. Therefore, if we pick a subsequence $\frac{x_{n_k} - x}{|x_{n_k} - x|} \to v \in \mathbb{S}^{N-1}$, then

$\frac{|f(x_n)-f(x)|}{|x_n-x|} \to (D_x f)(v)$. Now, since $v \in \mathbb{S}^{N-1}$, it is nonzero. Since $D_x f$ is injective, therefore $(D_x f)(v) \neq 0$. But by the above inequality,

$$(D_x f)(v) = \lim_{k\to\infty} \frac{x_{n_k} - x}{|x_{n_k} - x|} = 0$$

The latter limit follows since the sequence is squeezed between 0 and $\frac{1}{n}$.

This gives the required contradiction.

For the second result, let f also be C^1 in a neighbourhood of x. Now, we can assume for a contradiction, that there exist a sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ such that $x_n y_n \in B_{\frac{1}{n}}(x) \cap (x+V)$ and such that

$$\forall n \in \mathbb{N} : \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} < \frac{1}{n}$$

Notice that both $x_n \to x$ and $y_n \to x$. And, up to a subsequence, assume $\frac{|x_n-y_n|}{|x_n-y_n|} \to v \in \mathbb{S}^{N-1} \cap V$.

Since f is of $C^1(U)$ for some open neighbourhood $U \ni x$, we have by Lagrange's Mean value theorem, a $u_n \in S_n$, where $S_n = x_n + [0,1](y_n - x_n)$ is the line segment joining x_n and y_n, such that:

$$f(x_n) - f(y_n) = \langle \nabla f(u_n), (x_n - y_n)\rangle \text{ implying } \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \left| \langle \nabla f(u_n), \frac{x_n - y_n}{|x_n - y_n|} \rangle \right|$$

Since $x_n - \to x$ and $y_n \to x$, we can use the triangle inequality to deduce that $u_n \to x$, and by continuity of $u, w \mapsto \langle \nabla f(u), w\rangle$ (since $u \mapsto \nabla f(u)$ is continuous by $C^1(U) \ni f$ and the differential is linear, so always continuous), it follows that we can take the limit $n \to \infty$ to get:

$$\lim_{n\to\infty} \langle \nabla f(u_n), \frac{x_n - y_n}{|x_n - y_n|} \rangle = \langle \nabla f(x), v\rangle$$

Therefore,

$$0 = \lim_{n\to\infty} \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \lim_{n\to\infty} \left| \langle \nabla f(u_n), \frac{|x_n - y_n|}{|x_n - y_n|} \rangle \right| = |\langle \nabla f(x), v\rangle|$$

Which again, contradicts injectivity of the differential on V, since $v \in \mathbb{S}^{N-1} \cap V$, which does not contain 0, while $D_x f_V v = 0$

□

Remark 2. *Note that the result of proposition 2 also writes as:*

$$|f^{-1}(x) - f^{-1}(p)| \leq \frac{1}{\delta}|p - x|, \quad \forall p \in f(B_\delta(x) \cap (x+V))$$

Proving that the inverse function is ***continuous****. In the case f is C^1, we have* ***uniform continuity****:*

$$|f^{-1}(q) - f^{-1}(p)| \leq \frac{1}{\delta}|p - q|, \quad \forall p, q \in f(B_\delta(x) \cap (x+V))$$

. In the case that $V = \mathbb{R}^N$ and $N = M$, we will later prove that f^{-1} is also differentiable.

Proposition 3. *If $\Omega \subset \mathbb{R}^N$ is open and $f : \Omega \to \mathbb{R}^M$ is differentiable at $x \in \Omega$, with $D_x f$ injective, and f is an homeomorphism at $S \subset \Omega$, then*

$$D_x f[T_x S] = T_{f(x)} f(S)$$

Proof. By contradiction! Suppose that $w \in T_{f(x)} f(S)$ such that w is not $D_x(v)$ for any $v \in T_x S$. Let $y_n \to f(x)$ and $\lambda_n \to 0$ such that $\frac{y_n - f(x)}{\lambda_n} \to w$. Then we can define $(x_n)_{n \in \mathbb{N}} \subset S$ through $x_n = f^{-1}(y_n)$ and by continuity of f^{-1} it follows $x_n \to x$. Now, we consider the sequence

$$\left(\frac{x_n - x}{\lambda_n} \right)_{n \in \mathbb{N}}$$

If we take a subsequence, then $\frac{y_n - f(x)}{\lambda_n} \to w$ and $y_n \to f(x)$ still holds, so we should **never be able to find a subsequence** such that $\frac{x_n - x}{\lambda_n} \to v$ for some v, which would then be an element of $T_x S$, proving the contrary.

In other words, since $\{ \frac{x_n - x}{\lambda_n} \}_{n \in \mathbb{N}} \subset \mathbb{R}^N$, where we have the Bolzano-Weierstrass theorem, the only option that is left is that the sequence is **unbounded**. Up to taking subsequences, we can assume therefore

$$\frac{|x_n - x|}{\lambda_n} > n$$

Now, by injectivity of the differential, we also have a $\delta > 0$ such that

$$\forall y \in B_\delta(x) : \delta |y - x| \leq |f(y) - f(x)|$$

Take $n \geq N$, for N sufficiently large that $B_{\frac{1}{N}}(x) \subset B_\delta(x)$. Then, for all $n \geq N$, we have:

$$\delta n < \frac{\delta |x_n - x|}{\lambda_n} \leq \frac{|f(x_n) - f(x)|}{\lambda_n}$$

And this finishes the proof, because we assumed that $\frac{|f(x_n) - f(x)|}{\lambda_n} \to |w|$, and this has now become impossible □

Remark 3. *The reason why, **in general**, $D_x f[T_x S] = T_{f(x)} f(S)$ **fails**, is exactly because we can find w's, which are the limits of $\left(\frac{y_n - f(x)}{\lambda_n} \right)_{n \in \mathbb{N}}$ such that, while we still (by homeomorphism property) have that $f^{-1}(y_n) \to x$, the sequence*

$$\left(\frac{f^{-1}(y_n) - x}{\lambda_n} \right)_{n \in \mathbb{N}}$$

does not converge, not even up to subsequence, which must mean that it is unbounded. This is what happened in the counterexample mentioned at the end of the first section:

$$f(x)=x^3:\mathbb{R}\to\mathbb{R},\ \textit{then}\ f^{-1}:x\mapsto\sqrt[3]{x}$$

.

$$D_0 f=0,\ \textit{hence}\ D_0 f[T_x S]=\{0\}\ \textit{for all}\ S\subset\mathbb{R},\ \textit{while}\ T_{f(x)}f(\mathbb{R})=\mathbb{R}$$

We can take any nonzero $w\in\mathbb{R}=T_{f(0)}\mathbb{R}=T_{f(0)}f(\mathbb{R})$, let it be the limit of the sequence

$$x_n=\frac{w}{n}\to 0,\ \lambda_n=\frac{1}{n^3}\to 0,\quad \frac{f(\frac{w}{n})-f(0)}{\frac{1}{n^3}}=\frac{(\frac{w}{n})^3-0^3}{\frac{1}{n^3}}\to w$$

We see then, that no matter what subsequence $k\mapsto n_k$ we pick:

$$\frac{x_{n_k}-0}{\lambda_{n_k}}=\frac{\frac{w}{n_k}}{\frac{1}{n_k^3}}\to\infty$$

*Due to non-injectivity of the differential, we can map **non-convergent** $\frac{x_n-x}{\lambda_n}$ to **convergent** $\frac{f(x_n)-f(x)}{\lambda_n}$. The big problem is then that some convergent $\frac{f(x_n)-f(x)}{\lambda_n}$ do not have convergent $\frac{x_n-x}{\lambda_n}$, leaving $D_x f[T_x S]\subset T_{f(x)}f(S)$ **strict.***

1.2 Surjective differential

To quote the notes,

*"When the differential of a map $f:\Omega\to\mathbb{R}^N$ is surjective at a point x it means that, locally, the map **sends points in all directions**."*

Proposition 4. *If $\Omega\subset\mathbb{R}^N$ is **open**, $f:\Omega\to\mathbb{R}^M$ **differentiable**, and*

$$\forall x\in\Omega: D_x f \textit{ is } \textbf{\textit{surjective}}$$

(in particular $N\le M$). Then

$$\forall x\in\Omega:\exists\delta>0: B_{\delta^2/2}(f(x))\subset f(B_\delta(x))$$

Proof. The proof is not very intuitive:

Fix an $x\in\Omega$. Let $V\subset\mathbb{R}^N$ be a linear subspace of dimension M such that $D_x f|_V$ is a linear isomorphism $V\cong\mathbb{R}^M$. Since $D_x f|_V$ is injective now, we can find a $\delta>0$ such that

$$\delta|x-y|\le|f(x)-f(y)|,\quad\text{for all} y\in\overline{B_\delta(x)}\cap(x+V)$$

Where we can take the closure by considering a $\delta'<\delta$ or by just stating that f is continuous, so if $y\in\overline{B_\delta(x)}\cap(x+V)$, then take a sequence $(y_n)_{n\in\mathbb{N}}\subset B_\delta(x)\cap(x+V)$ with $y_n\to y$ and take the limit in the above inequality to conclude that it still holds in the closure.

Okay, now let $y \in B_{\delta^2/2}(f(x))$ be **any** y. The goal is to show that $y \in f(B_\delta(x)$, so we need to find a $p \in B_\delta(x)$ such that $f(p) = y$. Consider the function:

$$z \mapsto |f(z) - y| \text{ on } \overline{B_\delta(x)} \cap (x + V)$$

It is continuous so should assume its minimum on this **compact set** (Weierstrass' Theorem), say in p.

We show that $p \in B_\delta(x) \cap (x + V)$, and later we show $f(p) = y$. Let's start at the beginning:

$$\delta|p - x| \leq |f(p) - f(x)| \leq |f(p) - y| + |f(x) - y| \leq 2|f(x) - y| < \delta^2$$

This clarifies the somewhat strange choice of the radius $\frac{\delta^2}{2}$.

Next, $f(p) = y$. Assume not, and let $w = y - f(p) \neq 0$, therefore $|w| > 0$, and since p was at minimum distance of y in $f(B_\delta(x) \cap (x+V))$ it follows that:

$$B_{|w|}(y) \cap f(B_\delta(x) \cap (x + V)) = \emptyset$$

And this implies that there can be no sequence $y_n \to f(p)$ through $f(B_\delta(x) \cap (x + V))$ such that $\frac{y_n - f(p)}{\lambda_n} \to w$, since w "points out of $f(B_\delta(x) \cap (x + V))$, so:

$$w \notin T_{f(p)}[f(B_\delta(x) \cap (x + V))]$$

By injectivity of the differential, we have $\delta|x - y| \leq |f(x) - f(y)|, \quad \text{for all} y \in \overline{B_\delta(x)} \cap (x + V)$, we get that f is a **homeomorphism** on $B_\delta(x) \cap (x + V)$, so it follows that:

$$T_{f(p)}[f(B_\delta(x) \cap (x + V))] = (D_p f)[T_p[B_\delta(x) \cap (x + V)]]$$

Since p is an internal point (here, we make fruitful use of the fact $p \in B_\delta(x) \cap (x + V)$), it follows

$$T_p[B_\delta(x) \cap (x + V)] = V, \text{ and by surjectivity, } (D_p f)(V) = \mathbb{R}^M$$

Which contradicts $w \notin T_{f(p)}[f(B_\delta(x) \cap (x + V))]$ in every possible way. □

Remark 4. *Homeomorphisms can* ***change*** *the* ***differential structure*** *of a set. For example,*

$$f(x) = \begin{cases} \frac{|x|_\infty}{|x|_2} x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Maps the boundary of the unit square Q, that is,

$$\partial Q = \partial([-1, 1] \times [-1, 1]) \textit{ to the unit circle } \mathbb{S}^1 = \partial B_1(0)$$

Plainly, it "maps a square to a circle". It is a homeomorphism, but the square has a ***corner****, and the circle does not. To make this more precise, we can argue* *that tangent planes to the circle are not mapped to tangent planes, but may be mapped to strict cones:*

$$T_{(1,1)}\partial Q = (-\infty,] \times \{1\} \cup \{1\} \times (-\infty, 1]$$

*This is a strict cone, **not** a linear space, whereas*

$$T_{f(1,1)}\partial B = T_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)}\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x - y = 0\} \textbf{\textit{ is}}.$$

*But if a homeomorphism is **differentiable** at x and $D_x f$ is **injective**, we can conclude that tangent vector spaces get mapped to tangent vector spaces (of equal dimension). You can calculate the partial derivatives of this particular f to see which of these assumptions fails to hold.*

1.3 Diffeomorphisms

We saw that in order to get an equality

$$T_{f(x)}f(S) = D_x f[T_x S]$$

when mapping a tangent cone to a subset S of the domain to a tangent cone of its image $f(S)$, we needed the premised of Proposition 3:

1. $f : \Omega \to \mathbb{R}^N$ is **differentiable**.
2. $f : \Omega \to \mathbb{R}^N$ is a **homeomorphism** at $S \subset \Omega$.
3. $D_x f$ is **injective**.

We now introduce *diffeomorphisms*, which form a subclass of homeomorphisms that satisfy the above three properties and more. The idea that, if both f and its first order partial derivatives are homeomorphisms, this provides easier properties to check, and leads to theorems about the characterization of local behaviour of these functions (in particular, the Inverse Function Theorem).

Definition 2. ***Diffeomorphisms***
*Let $\Omega \subset \mathbb{R}^N$ be **open**, and $f : \Omega \to \mathbb{R}^N$. Then f is called a **diffeomorphism** if it is a **homeomorphism** of class $C^1(Omega)$ and also $f^{-1} : f(\Omega) \to \Omega$ is of class C^1*

Proposition 5. *Let $\Omega \subset \mathbb{R}^N$ be **open** and $f : \Omega \to \mathbb{R}^N$ a **diffeomorphism**. Then*

(i) *f is a **homeomorphism**.*

(ii) *For all $x \in \Omega$:*

$$\det(D_x f) \neq 0$$

(iii) *For all $x \in \Omega$:*

$$D_{f(x)}f^{-1} = [D_x f]^{-1}$$

(iv) For all $x \in \Omega$ *and for all* $S \subset \Omega$ *:*

$$T_{f(x)}f(S) = D_x f[T_x S]$$

So in particular, if $T_x S$ *is a vector space, then* $T_{f(x)}f(S)$ *is.*

Proof. (i) Any diffeomorphism is a homeomorphism, according to the definition.

(ii),(iii) Let $x \in \Omega$. We use that $f^{-1}(f(x)) = x$ and that both f and f^{-1} are **differentiable**, to conclude by the chain rule:

$$[J_{f(x)}f^{-1}][J_x f] = \mathrm{Id}_N$$

Therefore, we conclude that $(Jf^{-1})(x)$ and $(Jf)(x)$ are invertible, and by homomorphism property of the determinant,

$$\det(J_x f^{-1})\det(J_x f) = 1$$

(iv) For each $x \in \Omega$ and $S \subset \Omega$, f is a homeomorphism at S with injective differential at x, and therefore we can directly apply proposition 3 to conclude.

□

Remark 5. *The domain and target space of diffeomorphisms have the same dimension, that is, if* $\Omega \subset \mathbb{R}^M$ *and* $f : \Omega \to \mathbb{R}^N$*, then* $M = N$*. We could have ommitted this as part of the definition, since by the chain rule, we would still have*

$$[J_{f(x)}f^{-1}][J_x f] = \mathrm{Id}_M$$

By which we would have to conclude that $(D_x f) : \mathbb{R}^M \to \mathbb{R}^N$ *is invertible, therefore* $M = N$*.*

Next, we give a **characterization** of global diffeomorphisms in terms pf **global properties**.

Theorem 1. *Let* $\Omega \subset \mathbb{R}^N$ *be* ***open****, and* f *a function* $f : \Omega \to \mathbb{R}$*. Then* f *is a* ***diffeomorphism on*** Ω *if and only if*

1. f *is of* ***class*** $C^1(\Omega)$*.*
2. f *is* ***injective on*** Ω
3. *For all* $x \in \Omega$, $\det(J_x f) \neq 0$

Proof. Since (i) follows from the definition and (ii) from injectivity of the differential on $\mathbb{R}^N \cap \Omega$, and (iii) from the previous proposition in this section, if suffices to show that (i), (ii), and (iii) imply that f is a homeomorphism on Ω and f^{-1} is $C^1(f(\Omega))$.

First, $f(\Omega)$ is open, since the differential $D_x f$ is surjective at every $x \in \Omega$.

f is continuous, i.e. $C^0(\Omega)$ because it is even $C^1(\Omega)$. f^{-1} can be defined on $f(\Omega)$ by injectivity of f.

Next, since $D_x f$ is injective on $V = \mathbb{R}^N \subset \mathbb{R}^N$, at each $x \in \Omega$, there is a $\delta > 0$ such that for all $y \in B_\delta(x)$,

$$\delta|x-y| \leq |f(x) - f(y)|$$

We can rewrite this: if $y \in f(\Omega)$, where $y = f(x)$, look at the open neighbourhood $f(B_\delta(x))$ of y. Because, for all $p \in f(B_\delta(x))$,

$$|f^{-1}(y) - f^{-1}(p)| \leq |y-p|$$

And in particular, it follows that f^{-1} is continuous.

Finally, we prove that f^{-1} is even **differentiable**. We use the characterization given in Chapter 5, and prove that there is an $N \times N$ matrix M such that for any $(y_n)_{n\in\mathbb{N}} \subset f(\Omega)$ with $y_n \to y \in \Omega$, and $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1)$ such that $\lambda_n \to 0$ and

$$\frac{x_n - x}{\lambda_n} \to w$$

We then have:

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y)}{\lambda_n} = Aw$$

Now, since we have to conform with the previous proposition, we know what A should equal: $[J_x f]^-1$.

Define $x_n = f^{-1}(y_n)$. Since f is a homeomorphism, $x_n \to x$ where $x = f^{-1}(y)$. We can without loss of generality assume that $y_n \neq y$ for all $n \in \mathbb{N}$. Namely, if this holds for finitely many n, we can discard the initial segment and argue "up to subsequence", while if this holds for infinitely many n, we conclude that $\frac{y_n - y}{\lambda_n} = 0$ for infinitely many $n \in \mathbb{N}$, so the latter sequence has to converge to $w = 0$, and there is nothing to prove: for any $A \in \mathbb{R}^{N\times N}$, $A0 = 0$.

Assuming this, we can follow Chapter 5 and write:

$$(D_x f)\left(\frac{x_n - x}{\lambda_n}\right) = \frac{f(x_n) - f(x)}{\lambda_n} \tag{1}$$

$$+ \frac{f(x_n) - f(x) - (D_x f)(x_n - x)}{|x_n - x|} \frac{|x_n - x|}{|f(x_n) - f(x)|} \frac{|f(x_n) - f(x)}{\lambda_n} \tag{2}$$

By injectivity of $D_x f$, obtain a $\delta > 0$ such that

$$\delta|x_n - x| \leq |f(x_n) - f(x)|$$

Since $x_n \to x$, let $N \in \mathbb{N}$ be sufficiently large that for all $n \geq \mathbb{N}$, we have $x_n \in B_\delta(x)$.

Then, we can see:

$$
\begin{aligned}
(D_x f)\left(\frac{x_n - x}{\lambda_n}\right) &\to (D_x f)(v) \text{ since } D_x f \text{ is linear hence continuous,}\\
\frac{f(x_n) - f(x)}{\lambda_n} &\to w \text{ by assumption,}\\
\frac{f(x_n) - f(x) - (D_x f)(x_n - x)}{|x_n - x|} &\to 0 \text{ by differentiability of} f \text{ at } x.\\
\frac{|x_n - x|}{|f(x_n) - f(x)|} &\leq \frac{1}{\delta} \text{ by injectivity of } D_x f,\\
\frac{|f(x_n) - f(x)}{\lambda_n} &\to |w| \text{ by assumption}
\end{aligned}
$$

Therefore,

$$\frac{|x_n - x|}{|f(x_n) - f(x)|}\frac{|f(x_n) - f(x)}{\lambda_n} \text{ is bounded while } \frac{f(x_n) - f(x) - (D_x f)(x_n - x)}{|x_n - x|} \to 0$$

So we conclude, taking the limit on both sides, that

$$(D_x f)(v) = w \text{ or on a standard basis, } (J_x f)(v) = w$$

By $\det(J_x f) \neq 0$, we can invert and obtain $w = [J_x f]^{-1}$. This immediately shows $(J_x f^{-1}) = [J_x f]^{-1}$.

Finally, we show that f^{-1} is C^1. This simply follows from the fact that the entries of $(J_x f^{-1})(x)$ are obtained by Cramer's rule:

$$[J_x f]^{-1} = \frac{1}{\det((J_x f)}(\operatorname{cof}(J_x f))^t$$

Where $\operatorname{cof}(A)$ denotes the **matrix of cofactors** of A.

□

We are now ready to prove the inverse function theorem, which gives a *local* characterization of diffeomorphism.

Theorem 2. ***Inverse Function Theorem***

Let $\Omega \subset \mathbb{R}^N$ open, $: \Omega \to \mathbb{R}^N$ function of class $C^1(\Omega)$. If there is a $x \in \Omega$ such that $\det(D_x f) \neq 0$*, then There is a $r > 0$ such that*

$$f|_{B_r(x)} \textit{ is a diffeomorphism}$$

Proof. Since f is of class $C^1(\Omega)$, and $\det(D_x f)$ is polynomial in the partial derivatives $\partial_{x_i} f$, we have that $x \mapsto \det(D_x f)$ is continuous. Therefore, we conclude that the preimage of a maximal open neighbourhood of $\det(D_x f)$ is open, i.e. There is a $R > 0$ s.t. $\det(D_y f) \neq 0 \quad \forall y \in B_R(y)$. Moreover, we have $D_x f$ is injectve, so we can find a $\delta > 0$ and $r_1 > 0$ s.t.

$$\delta|y - x| \leq |f(y) - f(x)| \quad \forall y \in B_{r_1}(x)$$

In particular, f is injective on $B_{r_1}(x)$. If we set $r = \min\{r_1, R\}$, then we get that f is $C^1(B_r(x))$, injective on $B_r(x)$ and with an injective derivative on $B_r(x)$. It then follows by the above proposition that $f|_{B_r(x)}$ is a diffeomorphism. $\square$

The usefullness lies in the situation where we apply a **change of coordinates**. When computing an integral or rewriting a differential equation in different coordinates, then at least locally these coordinates should define a diffeomorphism. The trivial example would be polar coordinates, where

$$f(r, \theta) = (r\cos\theta, r\sin\theta)$$

Which is a diffeomorphism from $(0, \infty) \times (0, 2\pi) \to \mathbb{R}^2 \backslash \{x \geq 0, y = 0\}$

2 Homework

See handwritten solutions (I was too lazy to typeset it in LaTeX).