Analysis 2, Chapter 7

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1 Vector fields, forms and gradients

We investigate, under which conditions a **vector field** $V : \Omega \to \mathbb{R}^N$, $\Omega \subset \mathbb{R}^N$ is the **gradient** of a **scalar function** $\varphi : \Omega \to \mathbb{R}$ (i.e. **conservative**)?.

For continuous **scalar** vector fields $f : I \to \mathbb{R}$, where I is an interval, this is trivially true, since we can integrate it over an interval $[x, 0]$ or $[0, x]$, i.e. define

$$F(x) := \int_0^x f(x)dx$$

(Note: here, we set $\int_x^0 := -\int_0^x$ if $x < 0$). By the **fundamental theorem of calculus**, the derivative exists and equals f.

In general, we have seen counterexamples. For example:

$$V : (x, y) \mapsto (y^2, x^2); \mathbb{R}^2 \to \mathbb{R}^2$$

If this were a conservative vector field, $V = \nabla\varphi$, then we would have:

$$\begin{aligned} \partial_1\varphi(x, y) = y^2 &\implies \varphi(x, y) = xy^2 + g(y) \\ \partial_2\varphi(x, y) = x^2 &\implies \varphi(x, y) = yx^2 + h(x) \end{aligned}$$

For some functions $g, h : \mathbb{R} \to \mathbb{R}$. This leads to a contradiction. We see that V has to satisfy certain **algebraic** conditions.

There is also an interplay with **topology**, in the sense that there are necessary conditions on the **shape of the domain** Ω: this set **cannot have holes**. There is the counterexample:

$$V : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2$$
$$V(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

If V were conservative, it would integrate to 0 over any closed C^1- curve $\gamma : [a, b] \to \mathbb{R}^2$; because for $V = \nabla\varphi$, we have $\varphi \circ \gamma : [a, b] \to \mathbb{R}$ is differentiable, so by the fundamental theorem of calculus,

$$\int_a^b (\varphi \circ \gamma)'(t)dt = \varphi \circ \gamma(b) - \varphi \circ \gamma(a)$$

But, consider the C^1-curve $\gamma : [0, 2\pi] \to \mathbb{R}^2$ through $\gamma(t) = (\cos t, \sin t)$,:

$$\begin{aligned} 0 = \varphi \circ \gamma(2\pi) - \varphi \circ \gamma(0) &= \int_0^{2\pi} (\varphi \circ \gamma)'(t)dt \\ &= \int_0^{2\pi} \langle V(\gamma(t)), \gamma'(t) \rangle dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t)dt = 2\pi \end{aligned}$$

Which gives a contradiction.

1.1 Schwarz' theorem

This theorem gives a sufficient condition for when partial derivatives in diffent directions commute.

Theorem 1. ***Schwarz' theorem***
Let $f : \mathbb{R}^N \to R$ be of class $C^2(\mathbb{R}^N)$ (i.e. has continuous partial derivatives and continuous second-order partial derivatives $\partial_v \partial_w f$ for all $v, w \in \mathbb{R}^N$) in $\mathbb{R}^N$. Then,

$$\forall i, j = 1, ..., N : \partial_{e_i} \partial_{e_j} f = \partial_{e_i} \partial_{e_j} f$$

Remark 1. *The second order partial derivatives need to be continuous. Peano found the following counterexample if this is not true:*

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if}(x, y) \neq (0, 0) \\ 0 & \text{else} \end{cases}$$

To understand this, see that in polar coordinates, f writes as:

$$f(x, y) = \frac{1}{4} r^2 \sin(4\theta)$$

So f is continuous at 0. But notice that it oscillates when we let (x, y) circle the origin, and therefore f cannot be approximated with second derivatives (i.e. as a paraboloid). When we calculate the second order partial derivatives, we see that ∂_x and ∂_y do not commute: this Homework 7, Exercise 1.

Definition 1. *$f : \mathbb{R}^N \to \mathbb{R}$ and $x \in \mathbb{R}^N$, with $\partial^2_{ij} f := \partial_i \partial_j f$ existent at every point. The Hessian of f at x is defined as:*

$$H_x f = \begin{pmatrix} \partial^2_{11} f(x) & ... & \partial^2_{1N} f(x) \\ ... & ... & ... \\ \partial^2_{N1} f(x) & ... & \partial^2_{NN} f(x) \end{pmatrix}$$

As a corrollary of Schwarz' theorem, for a $C^2(\mathbb{R}^N)$-function f, the hessian is $H_x f$ is symmetric at every point of its domain.

1.2 Differential forms

A different look at the differential $D_x f$ of a real-valued function f on an open $S \subset \mathbb{R}^N$ at a point x. In general, we like to think of S as a **manifold**, although we will define manifolds after Chapter 8. On manifolds, we only have a notion of a directional derivative in directions $v \in T_x S$. But note that for **open** $S \subset \mathbb{R}^N$, $T_x S = \mathbb{R}^N$, so directional derivatives can be taken in whatever direction $v \in \mathbb{R}^N$ we want to. Let $f : S \to \mathbb{R}$ be $C^1(M)$. We now denote the differential slightly differently, namely as:

$$df(x) := D_x f$$

Then at every point $x \in S$, $df(x)$ is linear map in $(T_x S)^* = (\mathbb{R}^N)^*$. Therefore

$$df : S \to (\mathbb{R}^N)^*$$

If we equip $(\mathbb{R}^N)^*$ with the operator norm $|\cdot|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})}$, defined as:

$$|L|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})} = \max_{x \in \mathbb{S}^{N-1}} |Lx|$$

Where $\mathbb{S}^{N-1}$ is defined to be the unit sphere according to the ordinary Euclidean norm. In this setting, $df(\cdot) : x \mapsto D_x f$ is even continuous $T_x S \to (T_x S)^*$ with respect to the norms $|\cdot|_{\mathbb{R}^N}$, $|\cdot|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R})}$. We will now consider a class of functions that this generalizes to: the **1-form**. The proper definition of 1-forms requires some differential geometry, which we will *sketch* for now.

Definition 2. *Let $S \subset \mathbb{R}^N$ be an open subset. A map $\omega : S \to (T_x S)^*$ is called a **1-form** on S. The **vector space** of all 1-forms on S is denoted $\bigwedge^1(S)$*

Definition 3. ***A more differential geometric definition of a 1-form.***

*Let $S \subset \mathbb{R}^N$ be a **differentiable manifold** (roughly (and untruthfully) speaking, the image set of a diffeomorphism (Chapter 8) $\varphi : \Omega \to S$ where $\Omega \subset \mathbb{R}^M$ is **open**).*

*We define the **tangent bundle** TS of S to be the **union** of **all tangent cones** (actually, these are now hyperplanes, see Chapter 6 for a proof of this) **at all points** $x \in S$:*

$$TS = \bigcup_{x \in S} T_x S$$

*Since S is diffeomorphic to a $\Omega \subset \mathbb{R}^M$ **open** (in our definition of a manifold), we have $T_x S = (D_x \varphi)(\varphi(T_x \Omega))$ (again, see Chapter 8).*

*Then a map $\omega : TS \to \mathbb{R}$ is called a **1-form** on S, if its restriction to any **fibre** $T_x S$ is linear: this means that, for all $x \in S$:*

$$\omega_x = \omega|_{T_x S} : T_x S \to \mathbb{R} \textit{ it linear.}$$

Remark 2. *Suggesting the existence of a structure $\bigwedge^k(S)$. Discussing this would go too far beyond the theory and goals of Chapter 7, and requires the definition of tensors, etc. We will just regard $\bigwedge^1(S)$ as defined in Definition 2.*

Definition 4. ***The Vector Space Structure of*** $\bigwedge^1(S)$

On $\bigwedge^1(S)$ there is indeed a structure of addition and scalar multiplication with scalars in $\mathbb{R}$, making this into a vector space:

$$\begin{aligned} \omega + \xi \text{ through } (\omega + \xi)(x) &= \omega(x) + \xi(x) \text{ in } (\mathbb{R}^N)^*, \text{ for all } x \in S \\ \lambda\omega \text{ through } (\lambda\omega)(x) &= \lambda \cdot \omega(x) \text{ in } (\mathbb{R}^N)^*, \text{ for all } x \in S \end{aligned}$$

This is nothing more than recognizing that $\bigwedge^1(\mathbb{R}^N)$ is the space of vector fields $S \to (\mathbb{R}^N)^*)$, where $(\mathbb{R}^N)^* = \mathcal{L}(\mathbb{R}^N, \mathbb{R})$ is a linear space, so we can define addition and scalar multiplications pointwise on S. In fact, $\bigwedge^1(S)$ is just the space of ***co-vector fields*** *on S.*

Any vector field $V : S \to \mathbb{R}^N$ can be written as $V = \sum i = 1^N V_i e_i$ where $\{e^1, ..., e^N\}$ is the standard basis $\{e_1, ..., e_N\}$ or $\mathbb{R}^N$, and where $V_i : S \to \mathbb{R}$ is a function. This is because for each $x \in S$, we can write $V(x) \in \mathbb{R}^N$ uniquely as a linear combination $\sum i = 1^N V_i(x) e_i$ where $V_i(x) \in \mathbb{R}$

Likewise, any $\omega \in \Lambda^1(M)$ can be written as $\omega = \sum i = 1^N \omega_i e^i$, where $\{e^1, ..., e^N\}$ is the **dual** basis of $(\mathbb{R}^N)^*$ corresponding to the standard basis $\{e_1, ..., e_N\}$.

Notice that if ω is a 1-form, then $\omega(x) \in (\mathbb{R}^N)^*$ for every $x \in \mathbb{R}^N$ and $\omega(x) = \sum i = 1^N \omega_i(x) e^i$ where $\omega_i(x)$ is just a scalar, of course.

Definition 5. *We define $x_1, ..., x_N \in (\mathbb{R}^N)^*$ just as: x_i is the* ***projection on the i-th coordinate*** *$x_1 : v \mapsto v^i$, we already know this as the dual basis$\{e^1, ..., e^N\}$ of the standard basis $\{e_1, ..., e_N\}$. We then define $dx_1, ..., dx_N \in \bigwedge^1(S)$ as:*

$$dx_i(y) = x_i; \quad dx_i : S \to (\mathbb{R}^N \to \mathbb{R})$$

In Haskell: $dx_i :=$ flip const x_i

Lemma 1. *These elementary forms form a basis of $\bigwedge(S)$, when we regard $\bigwedge^1(S)$ as a* ***module*** *over the* ***function ring*** *$\mathcal{F}(S, \mathbb{R}) = \{f : S \to \mathbb{R}\}$, in other words, for all $\omega \in \bigwedge^1(S)$, we can find unique $\omega_1, ..., \omega_N : S \to \mathbb{R}$ such that:*

$$\forall x \in S : \omega(x) = \sum_{i=1}^{N} \omega_i(x) dx_i(x)$$

We already saw this in different notation when we said ω could be written as:

$$\omega(x) = \sum_{i=1}^{N} \omega_i(x) e^i$$

The only problem with this notation is that in the equation $\omega = \sum_{i=1}^N \omega_i e^i$, the type of e^i and $\bigwedge^1(S)$ does not match.

Remark 3. *In particular, if $f : \Omega \to \mathbb{R}^N$ **differentiable**, then $df \in \bigwedge^1(\Omega)$ and for any $x \in \Omega$ we can write*

$$df(x) = \sum_{i=1}^{N} \partial_i \cdot f(x) dx_i(x)$$

Giving, as vectors in $\bigwedge^1(\Omega)$, the equality:

$$df = \sum_{i=1}^{N} \partial_i f \cdot dx_i$$

Definition 6. *A 1-form $\omega \in \bigwedge^1(S)$ is said to be of $C^1(S)$ (or **continuous**, or **differentiable**) if each of its components $\omega_i : S \to \mathbb{R}$ is $C^1(S)$ (or **continuous**, or **differentiable**).*

*That is, every property of the 1-form is translated to a **property of its components.***

Definition 7. *A 1-form $\omega \in \bigwedge^1(S)$ is said to be **exact** if there is a $f \in C^1(\mathbb{R}^N)$ such that*

$$df = \omega$$

*Notice that at least for the definition, we require the domain of f to be $\mathbb{R}^N$. The **Poincaré lemma** will hold on more exotic domains, as long as they are **star-shaped** (more on this later).*

1.3 Poincaré Lemma

Definition 8. *Let $\omega \in \bigwedge^1(S)$ be **continuous** and let $\gamma : [0,1] \to S$ be piecewise C^1 (we also write $\gamma \in C^1_p([0,1])$, see **Homework 3, Exercise 2**). We define te integral of ω along γ as:*

$$\int_\gamma \omega = \int_0^1 \omega(\gamma(t))(\gamma'(t))dt$$

Where the right-hand side is just an ordinary Riemann integral (it is well-defined because of the regularity conditions on γ and ω).

Lemma 2. *If $\omega = df$ for an $f : \Omega \to \mathbb{R}$ differentiable, this writes as*

$$\int_\gamma df = \int_0^1 \langle \nabla f(\gamma(t)), \gamma'(t) \rangle dt$$

And the chain rule gives that this equals:

$$... = \int_0^1 (f \circ \gamma)'(t) dt = f(\gamma(1)) - f(\gamma(0))$$

Theorem 2. *Let $\omega \in \bigwedge^1(S)$ be **continuous** and $\gamma, \mu \in C^1_p([0,1])$. Then the following are equivalent:*

(i) For all $\gamma \in C^1_p([0,1])$, such that $\gamma(0) = \gamma(1)$, we have

$$\int_\gamma \omega = 0$$

(ii) For all $\gamma, \mu \in C^1_p([0,1])$, such that $\gamma(0) = \mu(0)$ and $\gamma(1) = \mu(1)$, we have

$$\int_\gamma \omega = \int_\mu \omega$$

*(iii) ω is **exact**.*

Proof. For (i) $\implies$ (ii), we use that if $\gamma(0) = \mu(0)$ and $\gamma(1) = \mu(1)$, then we can define the new $C^1_p([0,1])$-curve κ through:

$$\kappa(t) := \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \mu(1 - 2(t - \frac{1}{2})) & t \in [\frac{1}{2}, 1] \end{cases}$$

We traverse γ forwards and μ backwards. Since $(2\cdot)$, $(1-)$ and $(-\frac{1}{2})$ are $\mathbb{C}^1$, κ is $C^1_p([0,1])$ as their composition, and moreover it is indeed continuous at $\frac{1}{2}$ because $\gamma(1) = \mu(1)$. Finally, $\kappa(0) = \gamma(0) = \mu(0)$, so κ is closed, therefore by (i):

$$0 = \int_\kappa \omega$$

The right hand side is, by definition, can be rewritten as:

$$\begin{aligned} ... &= \int_0^{\frac{1}{2}} \omega(\gamma(2t))(2\gamma'(2t))dt + \int_{\frac{1}{2}}^1 \omega(\mu(1 - 2(t - \frac{1}{2})))(\mu'(1 - 2(t - \frac{1}{2}))) \cdot 2dt \\ &= \int_{\frac{1}{2}}^1 \omega(\mu(1 - 2(t - \frac{1}{2})))(\mu'(1 - 2(t - \frac{1}{2})))(-2)dt \end{aligned}$$

Where the final equality uses linearity of $\omega(x)$, for all x. Now substitute $u(t) = 2t$ in the first, and $v(t) = 2t - 2$ in the second integral:

$$... = \int_0^1 \omega(\gamma(u))(u\gamma'(u))du + \int_1^0 \omega(\mu(v)(\mu'(v))dv = \int_\gamma \omega - \int_\mu \omega$$

We conclude. For (ii) $\implies$ (iii), we **define** a function $f : \mathbb{R}^N \to \mathbb{R}$ through

$$f(x) := \int_0^1 \omega(tx)(x)dt = \int_\gamma \omega, \text{ for any } \gamma \in C^1_p([0,1]), \text{ by (ii)}$$

We need to show that $\partial_v f(x) = \omega(x)(v)$ for all v in $\mathbb{R}^N$ and all $x \in \mathbb{R}^N$. Note that

$$\partial_v f(x) = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

To compute this fraction we first find an expression of $f(x+hv)$, by connecting $x+hv$ to the origin with two segments:

$$S_1 = [0,1]x, \quad S_2 = x + [0,h]v$$

Both are parametrizable as $C^1([0,1])$-curves, and we skip the definition to argue that

$$f(x+hv) = \int_0^1 \omega(sx)(x)ds + \int_0^1 \omega(x+shv)(hv)ds$$

With $f(x) = \int_0^1 \omega(sx)(x)ds$, this yields (using linearity of $\omega(y)(\cdot)$ in the second equality):

$$\begin{aligned} \frac{f(x+hv)-f(x)}{h} &= \frac{1}{h}\int_0^1 \omega(x+shv)(hv)ds \\ &= \int_0^1 \omega(x+shv)(v)ds \\ &= \frac{1}{h}\int_0^h \omega(x+rv)(v)dr \end{aligned}$$

Then, by **continuity** of $\omega(\cdot)(v)$ **in** x, for every $\epsilon > 0$ there is a $\delta > 0$ such that if $|rv| = |x+rv-x| < \delta$, then:

$$|\omega(x+rv)(v) - \omega(x)(v)| < \epsilon$$

Meaning that, for $|h| < \frac{\delta}{|v|}$:

$$\begin{aligned} \left|\omega(x)(v) - \frac{1}{h}\int_0^h \omega(x+rv)(v)ds\right| &\leq \frac{1}{|h|}\int_0^1 |\omega(x)(v) - \omega(x+rv)(v)|dr \\ &\leq \frac{1}{|h|} \cdot |h| \cdot \sup_{r\in[0,h]} |\omega(x)(v) - \omega(x+rv)(v)| \\ &\leq \epsilon \end{aligned}$$

Proving (ii) $\implies$ (iii), namely:

$$\lim_{h\to 0} \frac{f(x+hv)-f(x)}{h} = \omega(x)(v)$$

For (iii) $\implies$ (i), we simply apply Lemma 2 and conclude.

□

Definition 9. *A $C^1(S)$* ***1-form*** *$\omega \in \bigwedge^1(S)$ is* ***closed*** *if for all $i,j \in [N]$, we have*

$$\partial_i \omega_j = \partial_j \omega_i$$

Definition 10. *A set $S \subset V$ where V is a $\mathbb{R}$-**vector space**, is **star-shaped** if there exists a $y \in S$ such that*

$$\forall x \in S : x + [0,1](y - x) \subset S$$

Remark 4. *Any convex set C has, **by definition***

$$\forall x, y \in C : x + [0,1](y - x) \subset C$$

As a consequence, any convex set is star-shaped too.

Lemma 3. ***Poincaré's Lemma***
*Let $\Omega \subset \mathbb{R}^N$ be **star-shaped** and let $\omega \in \bigwedge^1(\Omega)$ be of $C^1(\Omega)$. Then ω is **exact** if and only if it is **closed**.*

Remark 5. *We can be more general: the equivalence between **exactness** and **closedness** holds on any Ω that **has no holes in it**: the precise topological definition of this will come in the course Topology.*

As a corollary, we can restrict ω to a star-shaped neigbourhood if Ω is open, because open balls are trivially star-shaped (even convex):

Corollary 1. *Let $\Omega \subset \mathbb{R}^N$ be **open** and let $\omega \in \bigwedge^1(\Omega)$ be of $C^1(\Omega)$. Then there exists an $\epsilon > 0$ for which ω is **exact** in a ball $B_\epsilon(x)$, if and only if it is **closed** in a ball $B_\delta(x)$.*

2 Homework

See handwritten solutions (I was too lazy to typeset it in LaTeX).