Analysis 2, Chapter 6

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1 Tangent cone and differentiability

1.1 The Tangent Cone

Definition 1. *Let $M \geq 1$, $E \subset \mathbb{R}^M$ and $x \in \mathbb{R}^M$. Then we say a vector $v \in \mathbb{R}^M$ is **tangent to** E in x if there is a sequence $(x_k)_k \subset E$ with:*

$$\lim_{k\to\infty} \frac{x_k - x}{|x_k - x|} = \frac{v}{|v|}$$

or $v = 0$.

Remark 1. *This is equivalent to: there is a sequence $(x_k)_k \subset E$ and a sequence $(\lambda_k) \subset (0, 1)$ with $\frac{x_k - x}{\lambda_k} \to v$*

Definition 2. *We define the **tangent cone** T_xE **to** E **at** x to be*

$$T_xE = \{v \in \mathbb{R}^M : v \text{ tangent to } E \text{ at } x\}$$

Proposition 1. *The tangent cone T_xE is a **cone**, that is a set such that: if $v \in T_xE$ then $\forall \lambda \geq 0 : \lambda v \in T_xE$*

The tangent cone to a graph has various other properties, provided that the graph comes from a differentiable function. In particular, we will show that a function $f : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$ if and only if the **tangent cone to the graph** of f at $(x, f(x)) \in \Omega \times \mathbb{R}$ is a **non-vertical linear space**, meaning it is the image of a linear map (which linear map could that be...).

The notes use the following notation: Denote the **graph** $\{(x, f(x)) : x \in \Omega\}$ of f with graph(f). But given that we define functions as relations, a function **is its graph**, as a set. So, to ease notation, we first repeat the definition of a function in set theory:

Definition 3. *A function $A \to B$ is a **relation**, i.e. a $f \subset A \times B$ such that:*

$$\forall x \in A : \exists! y \in B : x \ f \ y$$

We denote this unique y with $f(x)$. In this way, $\text{graph}(f) = f$.

Theorem 1. *Suppose $\Omega \subset \mathbb{R}^N$ is an* ***open*** *set and $x \in \Omega$, and $f : \Omega \to \mathbb{R}$ and $w \in (\mathbb{R}^N)^*$. Then* ***the following are equivalent****:*

(i) f is ***differentiable*** *at x and $D_x f = L$*

(ii) If $v \in \mathbb{R}^N = T_x\Omega$ (since Ω is ***open****) where*

$$\lim_{k\to\infty} \frac{x_k - x}{\lambda_k} = v$$

then it holds:

$$\lim_{k\to\infty} \frac{f(x_k) - f(x)}{\lambda_k} = Lv$$

We can also write this in terms of the gradient:

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be an* ***open*** *set and $x \in \Omega$, and $f : \Omega \to \mathbb{R}$ and $w \in \mathbb{R}^N$. Then the following are equivalent:*

(i) f is ***differentiable*** *at x and $\nabla f(x) = w$*

(ii) If $v \in \mathbb{R}^N = T_x\Omega$, where

$$\lim_{k\to\infty} \frac{x_k - x}{\lambda_k}$$

then it holds:

$$\lim_{k\to\infty} \frac{f(x_k) - f(x)}{\lambda_k} = \langle w, v\rangle$$

Proof. We show (i) $\implies$ (ii) and (ii) $\implies$ (i) separately:

(i) $\implies$ (ii) If $x_n = x$ only for **finitely many** $n \in \mathbb{N}$, then **eventually** (for $n \geq N$, N sufficiently large) $x_n \neq x$, and we have:

$$\frac{f(x_n) - f(x) - (D_x f)(x_n - x)}{|x_n - x|} \to 0$$

By **differentiability**. From this, we can rewrite, for $n \geq N$:

$$\frac{f(x_n) - f(x)}{\lambda_n} = \frac{f(x_n) - f(x) - (D_x f)(v)}{|x_n - x|} \frac{|x_n - x|}{\lambda_n} + (D_x f)\left(\frac{x_n - x}{\lambda_n}\right)$$

Since $\frac{x_n - x}{\lambda_n} \to v$, we have $\frac{|x_n - x|}{\lambda_n} \to |v| < \infty$, and by **continuity** of the differential map $D_x f$, we have

$$\frac{f(x_n) - f(x) - (D_x f)(v)}{|x_n - x|} \frac{|x_n - x|}{\lambda_n} + (D_x f)\left(\frac{x_n - x}{\lambda_n}\right) \to 0 \cdot |v| + (D_x f)(v)$$

which implies convergence of $\frac{f(x_n) - f(x)}{\lambda_n}$ to $(D_x f)(v)$. If $x_n = x$ for infinitely many $n \in \mathbb{N}$, we have to conclude:

$$\forall n \in \mathbb{N} : \exists m \geq n : \frac{x_m - x}{\lambda_m} \text{ , therefore necessarily } v = \lim_{n\to\infty} \frac{x_n - x}{\lambda_n} = 0 = 0$$

On the other hand, for the same m, we have

$$\frac{f(x_m)-f(x)}{\lambda_m}=0, \text{ , therefore necessarily } \lim_{n\to\infty}\frac{f(x_n)-f(x)}{\lambda_n}=0=(D_x f)(0)$$

So, in both cases, $\lim_{n\to\infty}\frac{f(x_n)-f(x)}{\lambda_n}=(D_x f)(v)$

(ii) $\implies$ (i) Let $(x_n)_{n\in\mathbb{N}}$ be any sequence with $x_n\to x$ as $n\to\infty$. We need to prove that the following sequence goes to 0 as $n\to\infty$:

$$\left(\frac{f(x_n)-f(x)-L(x_n-x)}{|x_n-x|}\right)_{n\in\mathbb{N}}$$

To prove this, we use the **Urysohn lemma**. Pick any **subsequence** of this sequence. We show that it has a **further subsequence** that **converges** to 0, thereby the result (i) follows.

Any subsequence $\left(\frac{f(x_{n_k})-f(x)-L(x_{n_k}-x)}{\lambda_{n_k}}\right)_{k\in\mathbb{N}}$ comes with sequences $(x_{n_k})_{k\in\mathbb{N}}$. Since $\frac{x_{n_k}-x}{|x_{n_k}-x|}\in\mathbb{S}^{N-1}$, a compact set, extract a subsequence labeled with n_{k_l} such that $\frac{x_{n_{k_l}}-x}{|x_{n_{k_l}}-x|}\to v\in\mathbb{S}^{N-1}$. Therefore, it follows by applying (ii) to $\lambda_l=|x_{n_{k_l}}-x|$ that:

$$\lim_{l\to\infty}\frac{f(x_{n_{k_l}})-f(x)-L(x_{n_{k_l}}-x)}{\lambda_{n_{k_l}}}=Lv$$

$$=L\left(\lim_{l\to\infty}\frac{x_{n_{k_l}}-x}{|x_{n_{k_l}}-x|}\right)=\lim_{l\to\infty}L\left(\frac{x_{n_{k_l}}-x}{|x_{n_{k_l}}-x|}\right)$$

Where the latter equality uses **continuity of linear maps**. Bringing the right limit to the left side, and taking the scalar division through the linear map, it follows:

$$\frac{f(x_{n_{k_l}})-f(x)-L(x_{n_{k_l}}-x)}{|x_{n_{k_l}}-x|}\to 0$$

Which was to be shown. The conclusion follows by the Urysohn property. □

Proposition 2. *Let $\Omega\subset\mathbb{R}^N$ **open**, $x\in\Omega$, $f:\Omega\to\mathbb{R}$ **differentiable** at x. Then $T_{(x,f(x))}f\subset\mathbb{R}^{N+1}$ is a **linear space**, and it has **dimension** N. In particular:*

$$T_{(x,f(x))}f=\left\{(v,\partial_v f(x)):v\in\mathbb{R}^N\right\}$$

*Which means that the space is generated by $\{(e_i,\partial_i f(x))\}_{i=1}^N$, because $(v,\partial_v f(x))+(w,\partial_w f(x))=(v+w,\partial_{v+w}f(x))$ by linearity of $v\mapsto\partial_v f(x)$, making it indeed N-**dimensional**.*

Proof. $\subset$ Let $(v, p) \in T_{(x,fx)}f$. We need to show $p = \partial_v f(x)$. Let $(x_n, y_n)_{n\in\mathbb{N}}$ be the sequence in f such that $x_n, y_n \to x, f(x)$, where this implies $y_n = f(x_n)$ and $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence so that

$$\frac{(x_n, f(x_n)) - (x, f(x))}{\lambda_n} \to (v, p)$$

Since this implies that we have separate convergence of $\frac{x_n - x}{\lambda_n} \to v$ and $\frac{f(x_n)) - f(x)}{\lambda_n} \to p$, while 1.1 implies that $\frac{f(x_n)) - f(x)}{\lambda_n} \to (D_x f)(v) = \partial_v f(x)$, it follows $p = \partial_v f(x)$.

$\supset$ **This inclusion does not require differentiability: we only need that $\partial_v f(x)$ exists.** Let $v \in \mathbb{R}^N$, then we need to show $(v, \partial_v f(x)) = \lim_{n\to\infty} \frac{(x_n, f(x_n)) - (x, f(x))}{\lambda_n}$ for some sequences $(\lambda_n)_{n\in\mathbb{N}} \subset (0, \infty)$, $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$. Let's take $x_n + \frac{1}{n}v$, and $\lambda_n = \frac{1}{n}$. Then since $\partial_v f(x)$ exists, it follows that

$$\lim_{n\to\infty} \frac{f(x + \frac{1}{n}v) - f(x)}{\frac{1}{n}} = \lim_{t\to 0} \frac{f(x + tv) - f(x)}{t} = \partial_v f(x)$$

Therefore,

$$\lim_{n\to\infty} \frac{(x_n, f(x_n)) - (x, f(x))}{\lambda_n} = (v, \partial_v f(x))$$

Therefore, we have shown $(v, \partial_v f(x)) \in T_{(x,fx)}f$, proving the second inclusion.

$\square$

Remark 2. *Note that this also writes as:*

$$T_{(x,fx)}f = \left\{(v, D_x f(v)) : v \in \mathbb{R}^N\right\} = D_x f$$

This is suggestive notation! Just imagine what we can do for $f : \Omega \to \mathbb{R}^M$...

Note that in 2, we only used that $\partial_v f(x)$ existed in order to prove the inclusion $\supset$. This leads to the following weaker proposition (which I will call a corrolary due to the proof already given):

Corrolary 1. *if $\Omega \subset \mathbb{R}^N$ is* ***open*** *and $f : \Omega \to \mathbb{R}$ only has, for a certain $v \in \mathbb{R}^N$, a* ***partial derivative*** *$\partial_v f(x)$ at $x \in \Omega$, then*

$$(v, \partial_v f(x)) \in T_{(x,fx)}f$$

In principle, the tangent cone to the graph of f at the point $(x, f(x))$ always contains the partial derivatives $\partial_v f(x)$ of f, provided that they exist and provided that Ω is **open**. In that case, namely, we can approach x **along a straight line**, say with a sequence $x_n = x + \frac{v}{n}$, $\lambda_n = \frac{1}{n}$ therefore $\lim_{n\to\infty} \frac{f(x_n) - f(x)}{\frac{1}{n}} = \lim t \to 0 \frac{f(x+tv) - f(x)}{t} = \partial_v f(x)$. However:

(i) If Ω is **not open**, not all v may be the limit of a sequence $\frac{x+t_n v-x}{t_n}$, that is, of a sequence $(x_n)_{n\in\mathbb{N}}$ lying on a straight line.

(ii) In **any case**, existence of partial derivatives does **not give information** about the behaviour of

$$\left(\frac{f(x_n)-f(x)}{\lambda_n}\right)_{n\in\mathbb{N}}$$

for *arbitrary* sequences $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$, $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_n \to 0$, even if $\frac{x_n-x}{\lambda_n}$ converges to some $v \in T_x\mathrm{dom}(f)$. This is because we have **no information** about sequences that approach x along arbitrary curves. This is **unless** we know that f is **differentiable**, in which case we can use Theorem 1.1.

1.2 Characterization of Differentiability by the Tangent Hyperplane

From the preceding section, we conclude the following characterization of differentiability of scalar functions $f : \Omega \to \mathbb{R}$ (which may be easier to work with than the original definition of $\exists L \in (\mathbb{R}^N)^* : f(y) - f(x) - L(y-x) \in o(|y-x|)$)

Theorem 3. *Let $x \in \Omega$, $\Omega \subset \mathbb{R}^N$ an **open** set and $f : \Omega \to \mathbb{R}$. Then f is **differentiable at** x if and only if:*

*(i) f is **continuous** at x.*

*(ii) $\forall v \in \mathbb{R}^N : \partial_v f(x)$ **exists** and moreover $v \mapsto \partial_v f(x)$ is **linear**.*

*(iii) $T_{(x,fx)}f$ **is a linear space of dimension** N.*

Proof. That differentiability implies **continuity**, (i), is a result of **??** given in Chapter 5. That it implies (ii), is a result of the fact $\partial_v f(x) = (D_x f)(v)$. That it implies (iii), follows from 2 and the fact that the image of a linear map is a linear space, which is generated by $(e_i, \partial_i f(x))$, $i = 1, ..., N$.

The converse is a bit more involved, we can use the characterization given by 1.1. If we can prove premise 1.1(ii), we are done. Therefore, consider any $v \in \mathbb{R}^N$, where the sequence $(x_n)_{n\in\mathbb{N}}$ in Ω has, together with $(\lambda_n)_{n\in\mathbb{N}}$, the property

$$\frac{x_n - x}{\lambda_n} \to v$$

Then we need to show $\frac{f(x_n)-f(x)}{\lambda_n} \to Lv$, for some L that is a linear map that does not depend on our choice of v, $(x_n)_{n\in\mathbb{N}}$ or $(\lambda_n)_{n\in\mathbb{N}}$.

The opposite implication follows as follows: Since the partial derivatives exist and are linear, we can define a linear map $L : \mathbb{R}^N \to \mathbb{R}$ through:

$$Lv = \sum_{i=1}^{N} v^i \partial_i f(x)$$

We already have

$$\lim_{n\to\infty} \frac{f(x_n)-f(x)}{\lambda_n} = Lv$$

For all **straight** sequences, where $x_n = x + \lambda_n v$. We now need to show that

$$\lim_{n\to\infty} \frac{f(x_n)-f(x)-L(x_n-x)}{|x_n-x|} = 0$$

Holds for any sequence $(x_n)_{n\in\mathbb{N}}$, and this does **not** follow from (i) and (ii) alone.

Using (iii), we know that $T_{(x,fx)}f$ is a **linear space of dimension** N and **contains a linear space** $\{(v, \partial_v f(x)) : v \in \mathbb{R}^N\}$, which has dimension N. So we must have $T_{(x,fx)}f = L$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to x such that $\frac{x_n-x}{|x_n-x|} \to v$. How to show that $\frac{f(x_k)-f(x)}{|x_k-x|}$ converges? Urysohn property! Pick any subsequence $k \mapsto n_k$ and relabel $x_k = x_{n_k}$, to ease notation. Either, we can pick a further subsequence

$$\frac{f(x_{k_l})-f(x)}{|x_{k_l}-x|}$$

that converges, or not. Assume this cannot be done. Since we can always extract a monotone subsequence, this must imply that $\frac{f(x_n)-f(x)}{|x_n-x|}$ is unbounded, otherwise we can extract a monotone, bounded subsequence, which would be convergent in $\mathbb{R}$. Up to a subsequence, we can therefore assume

$$\frac{|f(x_k)-f(x)|}{|x_k-x|} > k$$
$$\implies \frac{|x_k-x|}{|f(x_k)-f(x)|} \to 0$$

Moreover, using that $\frac{f(x_k)-f(x)}{|f(x_k)-f(x)|} \in \mathbb{S}^{N-1}$, we use Bolzano-Weierstrass to assume up to a subsequence that

$$\frac{f(x_k)-f(x)}{|f(x_k)-f(x)|} \to w \in \mathbb{S}^{N-1}$$

So using $\mu_k := |f(x_k)-f(x)|$, we have

$$x_k \to x, \frac{x_k-x}{\mu_k} \to 0, \frac{f(x_k)-f(x)}{\mu_k} \to w \neq 0, \text{ since } |w| = 1.$$

Therefore, $(0,w) \in T_{(x,fx)}f$, for some $w \neq 0$, which contradicts the assumption that $T_{(x,fx)} = L$, since $L0 = 0$. So we conclude that we can always pick a further converging subsequence $\frac{f(x_{k_l})-f(x)}{|x_{k_l}-x|}$, that converges to say w, for which by premise (iii), we have $w = Lv$, and therefore $\frac{f(x_n)-f(x)}{|x_{k_l}-x|} \to Lv$. This means that 1.1(ii) holds, and by 1.1[(ii) $\implies$ (i)], it follows that f is differentiable at x.
□

The following argument for the implication (i),(ii),(iii) $\implies$ **differentiability**, is **wrong**:

By continuity, $f(x_n) \to f(x)$, so that $(x_n, f(x_n))_{n\in\mathbb{N}}$ converges to $(x, f(x))$ through the graph f. Therefore, if $\frac{f(x_n)-f(x)}{\lambda_n}$ converges to some $p \in \mathbb{R}$, this would imply $p \in T_{(x,fx)}f$. Notice that since the partial derivative exists, we have that $\frac{f(x_n)-f(x)}{\lambda_n}$ indeed converges, and to $\partial_v f(x)$. So the only thing left is to show that these partial derivatives can be "united" to a linear map L.

For this, consider $v = e_i$, for $i = 1, ..., N$. Then, we can define a linear map $L : \mathbb{R}^N \to \mathbb{R}$ through

$$Lv := \sum_{i=1}^{N} v^i \partial_i f(x)$$

, where $v = \sum_{i=1}^N v^i e_i$. This defines a unique linear map since we fix L on a generating set of $\mathbb{R}^N$. It is also well-defined since we fix L on a linearly independent set of $\mathbb{R}^N$. It remains to show that for any sequence $(x_n)_{n\in\mathbb{N}}$ such that $\frac{x_n-x}{\lambda_n} \to v$, we have $\frac{f(x_n)-f(x)}{\lambda_n} \to Lv$. We see this, because

$$\frac{f(x_n) - f(x)}{\lambda_n} \to \partial_v f(x) = \sum_{i=1}^{N} v^i \partial_i f(x) \text{ by linearity of } v \mapsto \partial_v f(x),$$

$$\text{and } \sum_{i=1}^{N} v^i \partial_i f(x) = Lv \text{ by definition}$$

Therefore, the premise 1.1(ii) holds, and it follows that f is differentiable at x with $D_x f = L$.

The essential mistake is that we assume to have information about $\frac{f(x_n)-f(x)}{\lambda_n}$ for arbitrary sequences $(x_n)_{n\in\mathbb{N}} \subset \Omega$: $x_n \to x$, $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1)$: $\lambda_n \to 0$, *just because we know* $\lim n \to \infty \frac{f(x+t_n v)-f(x)}{t_n} = \partial_v f(x)$ for infinitesimal sequences $t_n \to 0$. It is true that for all $v \in \mathbb{R}^N$ we have $\partial_v f(x) \in T_{(x,fx)}f$ since Ω is open, so there is a sequence $(x_n)_{n\in\mathbb{N}}$ on a straight line with $(\lambda_n)_{n\in\mathbb{N}}$ such that we can approximate v, and by continuity of f and the shape of this sequence, we know $\lim n \to \infty \frac{f(x+t_n v)-f(x)}{t_n} = \partial_v f(x)$. Therefore, $T_{(x,fx)}f$ **contains the linear space** $\{(v, \partial_v f(x)) : v \in \mathbb{R}^N\}$ (which is a linear space by linearity of $v \mapsto \partial_v f(x)$), but is **may contain more**: see **Exercise 2** for an example where a function f **continuous at** $(0,0)$ with **existing and linear partial derivatives** at $(0,0)$, has a tangent cone to its graph that is strictly larger than the linear space spanned by its partials, and is also not differentiable.

And as it turns out, another sufficient condition is for f to be continuous and $T_{(x,fx)}f$ to be "non-vertical". This is clear in one dimension: we want the tangent line to be of the form $\{(t, f'(x)t)\}_{t\in\mathbb{R}}$. In $\mathbb{R}^N \times \mathbb{R}$, this translates to *"$T_{(x,fx)}f$ must be parametrizable with a linear map L".*

Theorem 4. *$x \in \Omega \subset \mathbb{R}^N$ an open set and $f : \Omega \to \mathbb{R}$. Then f is **differentiable** with $D_x f = L$ at x if and only if:*

(i) f *is* ***continuous*** *at* x.

(ii) $T_{(x,fx)}f = L$ *for some* ***linear map*** $L \in (\mathbb{R}^N)^*$

In other words, if we think about functions as sets, then **differentiability** of f just means that f is **continuous** and **its tangent cone is a linear map**, in particular the tangent cone to the graph equals (the graph of) its derivative!

Proof. We already know from proposition 3 if f is differentiable, $T_{(x,f(x))}f = \{(v, \partial_v f(x)) : v \in \mathbb{R}^N\}$. And from Chapter 5, we know f is continuous.

The converse follows, since

First, the premises of 3 imply differentiability, which in turn implies $T_{(x,fx)}f = D_x f$ by 2, which proves 4(ii).

Second, the premises of 4 should imply differentiability. Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ $(\lambda_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that $x_n \to \frac{x_n - x}{\lambda_n} \to v$. By continuity of f, we have $f(x_n) \to f(x)$, therefore $(x_n, f(x_n))_{n\in\mathbb{N}}$ is a sequence converging to $(x, f(x))$ through f, therefore if we can show $\frac{f(x_n)-f(x)}{\lambda_n}$ converges, it follows $\frac{f(x_n)-f(x)}{\lambda_n} \to Lv$ by premise 4(ii), and differentiability follows by 1.1[(ii) $\implies$ (i)].

But how to show that $\frac{f(x_n)-f(x)}{\lambda_n}$ converges? Urysohn property! Pick any subsequence $k \mapsto n_k$ and relabel $x_k = x_{n_k}$, $\lambda_k = \lambda_{n_k}$ to ease notation. Either, we can pick a further subsequence

$$\frac{f(x_{k_l}) - f(x)}{\lambda_{k_l}}$$

that converges, or not. Assume this cannot be done. Since we can always extract a monotone subsequence, this must imply that $\frac{f(x_n)-f(x)}{\lambda_n}$ is unbounded, otherwise we can extract a monotone, bounded subsequence, which would be convergent in $\mathbb{R}$. Up to a subsequence, we can therefore assume

$$\frac{|f(x_k) - f(x)|}{\lambda_k} > k$$
$$\implies \frac{|x_k - x|}{|f(x_k) - f(x)|} = \frac{\lambda_k}{f(x_k) - f(x)|} \frac{|x_k - x|}{\lambda_k} \to 0 \cdot |v| = 0$$

Moreover, using that $\frac{f(x_k)-f(x)}{|f(x_k)-f(x)|} \in \mathbb{S}^{N-1}$, we use Bolzano-Weierstrass to assume up to a subsequence that

$$\frac{f(x_k) - f(x)}{|f(x_k) - f(x)|} \to w \in \mathbb{S}^{N-1}$$

So using $\mu_k := |f(x_k) - f(x)|$, we have

$$x_k \to x, \frac{x_k - x}{\mu_k} \to 0, \frac{f(x_k) - f(x)}{\mu_k} \to w \neq 0, \text{ since } |w| = 1.$$

Therefore, $(0, w) \in T_{(x,fx)}f$, which contradicts the assumption that $T_{(x,fx)} = L$, since $L0 = 0$. So we conclude that we can always pick a further converging subsequence $\frac{f(x_{k_l})-f(x)}{\lambda_{k_l}}$, that converges to say w, for which by premise (ii), we have $w = Lv$, and therefore $\frac{f(x_n)-f(x)}{\lambda_n} \to Lv$. This means that 1.1(ii) holds, and by 1.1[(ii) $\Longrightarrow$ (i)], it follows that f is differentiable at x.

$\square$

2 Homework

Exercise 1 This exercise is about Leibniz' Integral Rule from Chapter 5. We apply it to compute a *Gaussian integral.* Define $F : [0, \infty) \to \mathbb{R}$, through:

$$F(x) := \int_0^x e^{-t^2} dt$$

(i) Prove that the limit

$$\lim_{x \to \infty} F(x)$$

exists and is finite.

(ii) Now, we want to compute it. A first idea would be to use the change of variables$s = t/x$ (for $x > 0$), to write:

$$F(x) = \int_0^x e^{-t^2} dt = x \int_0^1 e^{-x^2t^2} dt$$

Now, the integrand is:

$$g(t, x) = e^{-x^2t^2} = -\partial_x \left(\frac{e^{x^2t^2}}{t^2} \right)$$

So if we would apply Leibniz' Integration Rule, we would get

$$\int_0^1 e^{-x^2t^2} dt = -\partial_x \int_0^1 \frac{e^{-x^2t^2}}{t^2} dt$$

But, the integrand is not defined at $x = 0$, in other words $\phi(t, x) = e^{-x^2t^2}$ is only defined on any compact interval $[\epsilon, b]$ if $\epsilon > 0$.

An idea is to consider

$$\int_\epsilon^x e^{-t^2} dt$$

And to send $\epsilon \downarrow 0$. Explain why this does not work.

(iii) Instead, to overcome the singularity, consider

$$\phi(x) := \int_0^1 \frac{e^{-x^2(1+t^2)}}{1 + t^2} dt$$

Prove that

$$\phi'(x) = -(F^2(x))'$$

(iv) Obtain

$$\phi(x) = \frac{\pi}{4} - F^2(x)$$

(v) Prove

$$\lim_{x\to\infty} \phi(x) = 0$$

(vi) Conclude

$$\lim_{x\to\infty} F(x) = \frac{\pi}{4}$$

Proof

(i) It suffices to show that F is monotone and bounded. To see that it is monotone, take $x \leq y$ and see that

$$F(y) - F(x) = \int_x^y e^{-t^2} dt \geq |x-y| \inf_{t\in[x,y]} e^{-t^2} \geq |x-y|e^{-x^2} \geq 0$$

Therefore

$$x \leq y \implies F(x) \leq F(y)$$

For boundedness, we simply see that:

$$\begin{cases} 0 \leq e^{-t^2} \leq e^0 = 1 & \text{for } t \leq 1 \\ 0 \leq e^{-t^2} \leq e^{-t} & \text{for } t \geq 1 \end{cases}$$

Therefore,

$$\forall x \in \mathbb{R} : 0 \leq \int_0^x e^{-t^2} dt \leq \int_0^1 1 dt + \int_1^x e^{-t} dt = 1 + 1 - e^{-x} \leq 2$$

This means that for any sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to \infty$ (i.e. growing arbitrarily large for $n \geq N$, N sufficiently large), we have that $(F(x_n))_{n\in\mathbb{N}}$ is bounded and increasing, hence convergent to its supremum. This means $F(x) \to \sup_{x\in\mathbb{R}} F(x)$.

(ii) We don't get away with puttin ϵ in place of 0, since the lower bound of the integral will now depend on x in a way that is not defined at $s = 0$ either; if we let $s = t/x$, then

$$\int_\epsilon^1 e^{-t^2} dt = x \int_{\epsilon/x}^1 e^{-s^2x^2} ds$$

This time, if we would like to apply Leibniz' Integration Rule to the integrand $g(t,x) = e^{-x^2t^2}$, we would need to apply variant II of the Proposition, since $x \to a(x) = \epsilon/x$ is a lower bound that **depends on** x. The problem is that a is **not continuous** in an open neighbourhood Ω of 0, so the rule **fails to apply**.

(iii) We apply Leibniz' Integration rule to the integrand:

$$\psi(t,x) = \frac{e^{-x^2(1+t^2)}}{1+t^2}$$

This $\psi : [0,1] \times \Omega \to \mathbb{R}$ (where $\Omega = \mathbb{R}$ is open) is continuous on the whole of of its domain, since $t \mapsto 1 + t^2$ is a polynomial without zeroes and $(x,t) \mapsto e^{-x^2(1+t^2)}$ is a composition of the continuous exp-function with the continuous polynomial $-x^2(1+t^2)$, so their quotient is also continuous.

Moreover, $\phi(t, \cdot)$ is differentiable for all t, and:

$$\partial_x \psi(t,x) = -2xe^{-x^2(1+t^2)}$$

This function is clearly continuous in $[0,1] \times \mathbb{R}$, meaning that we can apply Leibniz' Integration Rule at any x in the domain $\Omega = \mathbb{R}$:

$$\phi'(x) = \partial_x \phi(x) = \int_0^1 \partial_x \frac{e^{-x^2(1+t^2)}}{1+t^2} = -2x \int_0^1 e^{-x^2(1+t^2)} dt$$

If we look at F, the other hand, by the fundamental theorem of calculus, it is differentiable with $F'(x) = e^{-x^2}$, so that by the chain rule, we have that $-(F^2(x))$ is differentiable and:

$$\begin{aligned} -(F^2(x))' &= -2F(x)F'(x) \\ &= -2xe^{-x^2} \int_0^1 e^{-t^2} dt \\ &= -2x \int_0^1 e^{-x^2(1+t^2)} dt \\ &= \phi(x) \end{aligned}$$

Proving the desired equality.

(iv) Since ϕ is differentiable and its derivative ϕ' e continuous, hence ϕ' integrable over the compact interval $[0, x]$, we can use the fundamental theorem of calculus to obtain:

$$\begin{aligned} \phi(x) &= \phi(0) + \int_0^x \phi'(s) ds \\ &= \int_0^1 \frac{dt}{1+t^2} - \int_0^x (F^2(x))' dx \\ &= \arctan(t) \mid_0^1 - F^2(x) \\ &= \frac{\pi}{4} - 0 - F^2(x) = \frac{\pi}{4} - F^2(x) \end{aligned}$$

(v) Fix $x \in \mathbb{R}$. For the integrand of ϕ, $t \mapsto \psi(t,x)$, notice that it by its differentiability and compact domain $[0,1]$, only has a few candidate points where it may assume its maximum (it assumes this by continuity (Weierstrass)): either in 0, 1 or in a stationary point:

$$\partial_t \frac{e^{-x^2(1+t^2)}}{1+t^2} = -\frac{2te^{-x^2(1+t^2)}((t^2+1)x^2+1)}{(t^2+1)^2}, \text{ therefore}$$

$$\partial_t \phi(t,x) = 0 \implies t = 0 \text{ or } t^2 = -1 - \frac{1}{x^2}$$

This means that only 0 and 1 are candidate extrema, and we obtain

$$\phi(0,x) = e^{-x^2}, \quad \phi(1,x) = \frac{1}{2}e^{-2x^2}$$

$$\sup_{t\in[0,1]} \left| \frac{e^{-x^2(1+t^2)}}{1+t^2} - 0 \right| = \max\left\{ e^{-x^2}, \frac{1}{2}e^{-2x^2} \right\} \to 0 \text{ as } x \to \infty$$

This means that the integrand goes to $0 : [0,1] \to \mathbb{R}; t \mapsto 0$ **uniformly**, as $n \to \infty$ for any sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to \infty$. If a sequence of Riemann-integrable functions defined on a compact interval uniformly converge to a limit function, we can exchange limit and integrand, and therefore we conclude

$$\begin{aligned} \lim_{x\to\infty} \phi(x) &= \lim_{x\to\infty} \in_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2} dt \\ &= \int_0^1 \lim_{x\to\infty} \left(\frac{e^{-x^2(1+t^2)}}{1+t^2} \right) dt \\ &= \int_0^1 0 dt = 0 \end{aligned}$$

(v) Since we also know $F(x) \to y$ for some $y \in \mathbb{R}$, as $x \to \infty$, we can conclude, since $F(x)$ is monotone and positive, that $F(x) = \sqrt{F^2(x)}$, and therefore:

$$F(x) = \sqrt{\frac{\pi}{4} - \phi(x)}$$

And by continuity of $s \mapsto \sqrt{s}$, and the established fact ((i)) that $\lim_{x\to\infty} F(x)$ exists, we conclude:

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \sqrt{\frac{\pi}{4} - \phi(x)} = \sqrt{\frac{\pi}{4} - \lim_{x\to\infty} \phi(x)} = \sqrt{\frac{\pi}{4}} = \frac{1}{2}\sqrt{\pi}$$

Exercise 2 Consider the function $f : (-1, 1)^2 \to \mathbb{R}$ defined as

$$f(x, y) := \begin{cases} x & y = x^2 \\ 0 & \text{else} \end{cases}$$

Then:

(i) Prove that the tangent cone at the graph of f at the origin is the set

$$\{(x, y, 0) : (x, y) \in \mathbb{R}^2\} \cup \{(x, 0, x) : x \in \mathbb{R}\};$$

(ii) Show that this set is not a linear space.

Proof

(i) We have, for each $v \in \mathbb{R}^2$, $(v, \partial_v f(x)) \in T_{(x,fx)}f$ By the (second?) proposition above. Note that

$$\partial_v f(x) = \lim_{t \to 0} \frac{f((0,0) + tv) - f(0,0)}{t}$$

And this limit exists, since $f(x, y) \neq 0$ only on a non-straight curve through the origin, so along any line, $f(x, y) \neq 0$ at most in one point, and then it will remain 0. Therefore

$$\frac{f((0,0) + tv) - f(x)}{t} = \frac{0 - 0}{t} = 0$$

eventually, so:

$$\text{for all } v \in \mathbb{R}^2 : (v, 0) \in T_{((0,0),f(0,0))}f$$

Hence, all vectors with third component equal to 0 are tangent to the graph.

There is only one case left to consider: can third component of a tangent vector be nonzero? This is only possible if we have a sequence $(x_n)_{n \in \mathbb{N}}$ converging to $(0, 0)$, with $f(x_n)$ convergent, and $N \in \mathbb{N}$ such that $\forall n \geq N : f(x_n) \neq 0$ (and we have an infinitesimal sequence $(\lambda_n)_{n \in \mathbb{N}}$ with $\frac{(x_n, f(x_n) - (0,0,0)}{\lambda_n} \to v \neq 0$). This can only mean that $\forall n \geq N$ we have $x_n = (p_n, p_n^2)$, since $f(x) \neq 0$ only if the second component is the square of the first.

Then, we have $p_n \to 0$, because $x_n \to 0$, and $\frac{(p_n, p_n^2)}{\lambda_n} \to w$ for some $w \in \mathbb{R}^2$. This means that $p_n/\lambda_n \to w_1$ finite, but this implies $p_n^2/\lambda_n = p_n \cdot p_n/\lambda_n \to 0 \cdot w_1 = 0$ because of the product rule for limits. Moreover, $f(x_n)/\lambda_n = p_n \lambda_n \to w_1$ because $f(p_n, p_n^2) = p_n$. This means that any tangent vector that is not of the form $(v, \partial_v f(0, 0))$ has to be of the form $(x, 0, x)$. But, we have not shown yet that any $x \in \mathbb{R}$ is possible for this.

This is, however, true: we can, for example, take $p_n = \frac{x}{n}$ and $\lambda_n = \frac{1}{n}$, for any $x \in \mathbb{R}$, then $(p_n, p_n^2) \to (0,0)$,

$$\frac{f(p_n, p_n^2)}{\lambda_n} = \frac{x/n}{1/n} \to x, \text{ hence } \frac{(p_n, p_n^2, f(p_n, p_n^2) - (0,0,f(0,0))}{\lambda_n} \to (x,0,x)$$

For any $x \in \mathbb{R}$ there is such therefore such a tangent vector.

Notice that we have exhausted all possibilities for tangent vectors $(v, w) \in \mathbb{R}^2 \times \mathbb{R}$ by this case distinction on w. Therefore, we have also shown the other inclusion:

$$\{(x, y, 0) : (x, y) \in \mathbb{R}^2\} \cup \{(x, 0, x) : x \in \mathbb{R}\} \subset T_{(x,fx)} f$$

(ii) $(1,1,0)$ and $(1,0,1)$ are in this set, but $(1,1,0) + (1,0,1) = (2,1,1)$ is not of the required form. So the set is not closed under $+$, and therefore not a linear subspace.

Exercise 3 Let $\Omega \subset \mathbb{R}^N$ be an **open** set, and let $a \in \Omega$. Let $f : \Omega \to \mathbb{R}$ be differentiable at a. Prove that

$$T_{(a,f(a))} f = \left\{(v, \partial_v f(a)) : v \in \mathbb{R}^N\right\}$$

Proof Refer to Proposition 2.