Analysis 2, Chapter 4

Matthijs Muis

December 2023

1 Spaces of continuous functions

Definition 1. *For (X, d_X), (Y, d_Y) metric spaces, define the* ***space of bounded functions*** *$X \to Y$:*

$$B(X, Y) = \{f : X \to Y : \mathrm{diam}(f(X)) < \infty\}$$

Notation 1. *We will use the shorthand: $B(X) := B(X, \mathbb{R})$*

Note that $f \in B(X, Y)$ if and only if one of the following holds:

$$\forall y \in Y : \exists R > 0 : f(X) \subset B_R(y)$$
$$\exists y \in Y : \exists R > 0 : f(X) \subset B_R(y)$$

Definition 2. *We can equip $B(X, Y)$ with the* ***uniform metric****:*

$$d_\infty(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$$

Lemma 1. *$B(X, Y)$ with d_∞ is* ***closed****.*

Proof. Let $f_n \to f$ with $f_n \in B(X, Y)$. Like in Chapter 3, we argue with the following triangle inequality: for any $x, y \in X$,

$$d_Y(f(x), f(y)) \leq d_Y(f_n(x), f(x)) + d_Y(f_n(y), f(y)) + d_Y(f_n(x), f_n(y))$$

We can bound $d_Y(f_n(x), f(y))$ and $d_Y(f_n(y), f(y))$ by 1 if we let $n \geq N_1$, and we know by $f_n \in B(X, Y)$ that $d_Y(f_n(x), f_n(y))$ is bounded. So we have a bound for the right hand side, namely

$$\sup n \in \mathbb{N}\mathrm{diam}(f(X)) + 2$$

Therefore, conclude that $f \in B(X, Y)$. □

Lemma 2. *If Y is* ***complete****, then $B(X, Y)$ with d_∞ is* ***complete****. The converse is also true: if Y is* ***not complete****, then $B(X, Y)$ is* ***not complete*** *either (consider a sequence of* ***constant*** *functions $f_n = y_n$ where $(y_n)_{n \in \mathbb{N}} \subset Y$ is* ***Cauchy*** *but* ***not convergent****).*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be Cauchy. As a consequence, for every $x \in X$,

$$(f_n(x))_{n\in\mathbb{N}} \subset Y$$

is Caucy, and by completeness of Y, this sequence has a limit. Define $f : X \to Y$ through

$$f(x) := \lim_{n\to\infty} f_n(x), \quad x \in X$$

Left to show is that f is continuous and that $f_n \to f$ uniformly. The former follows from the latter. To show the first, we argue by continuity of the norm, that for any $z \in X$:

$$|f(z) - f_n(z)| = \lim_{m\to\infty} |f_m(z) - f_n(z)|$$

And $|f_m(z) - f_n(z)| < \epsilon$, $\forall z \in X$, if we let $m, n \geq N_\epsilon$. Therefore, if $n \geq N_\epsilon$, then

$$\sup_{x\in X} |f(x) - f_n(x)| = \sup_{x\in X} \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \sup_{x\in X} \epsilon \leq \epsilon$$

So we conclude $|f(x) - f_n(x)|_{C^0} \leq \epsilon$ for all $n \geq N\epsilon$. This implies the uniform convergence. □

Definition 3. *For (X, d_X), (Y, d_Y) metric spaces, define:*

$$C^0(X, Y) = \{f : X \to Y : f \text{ continuous}\}$$
$$C^0_b(X, Y) = C^0(X, Y) \cap B(X, Y)$$

Remark 1. *If X is* ***compact****, we have $C^0_b = C^0$ since continuous functions assume a* ***minimum*** *and a* ***maximum*** *on a compact set.*

Lemma 3. *$C^0(X, Y)$ and $C^0(X, Y)$ are* ***closed*** *with respect to d_∞.*

Proof. Let $f_n \to f$. A sequence of continuous functions converging uniformly has a continuous limit function, as we saw in Chapter 3. So $f \in C^0(X, Y)$.

A finite intersection of closed sets is closed, so $C^1_0(X, Y)$ is closed by closedness of $B(X, Y)$ and $C^0(X, Y)$. □

Lemma 4. *If (Y, d_Y) is* ***complete****, then $C^0(X, Y)$ and $C^0_b(X, Y)$ are also* ***complete*** *with respect to d_∞.*

Proof. We can argue the same way as in Lemma 2, since defining a limit $f(x) := \lim_{n\to\infty} f_n(x)$ relies only on completeness of Y. Then we prove uniform convergence, which does not need any of the properties of the underlying spaces. Finally, we can use closedness of $C^0(X, Y)$ and $C^0_b(X, Y)$ to conclude that the limit function lies within the said function space. □

1.1 Characterization of compact sets: the Ascoli-Arzelà theorem

Spaces of functions are often not finite-dimensional: Bolzano-Weierstrass is not a valid tool here. But recall that in complete metric spaces, compact sets are precisely the **closed** and **totally bounded** sets.

We know that if Y is **complete**, then the aforementioned $C^0(X,Y)$, $C_b^0(X,Y)$, $B(X,Y)$ are also complete with respect to d_∞. So we could use this characterization. But another characterization works better and is simpler to check for functions.

Definition 4. *A set of functions $\mathcal{K} \subset C^0(X,Y)$ is called **equicontinuous** if all $f \in \mathcal{K}$ admit a **same modulus of continuity** (implying that they are uniformly continuous). Formally:*

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in X : \forall s \in B_\delta(x) : f(s) \in B_\epsilon(f(x))$$

Remark 2. *We can translate this to a lower level of abstraction: a set $\mathcal{K} \subset C^0(X,Y)$ is called **equicontinuous** if for all $\epsilon > 0$ There is a $\delta > 0$ such that:*

$$\forall f \in \mathcal{K} : \forall x, y \in X : d_X(x,y) < \delta : d_Y(f(x), f(y)) < \epsilon$$

Theorem 1. ***Ascoli-Arzelà theorem***

*For (X, d_X), (Y, d_Y) metric spaces, where X is **compact** and Y is **complete**, $\mathcal{K} \subset C^0(X,Y)$ **compact** with respect to the uniform metric d_∞ ("with respect to uniform convergence") if and only if:*

(i) For all $x \in X$, we have that

$$\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\} \text{ is } \textbf{compact}$$

*(ii) $\mathcal{F}$ is **equicontinuous**.*

*(iii) $\mathcal{F}$ is **closed** with respect to d_∞*

Corrolary 1. *Let $(X, d1)$ be a **compact** metric space, and let $(Y, d2)$ be a **complete** metric space. Let $(f_n)n \in N \subset C^0(X,Y)$ be a sequence of **equicontinuous** functions such that, for each $x \in X$, the set*

$$\{f_n(x) : n \in \mathbb{N}\}$$

*is **compact**. Then, there exists a **subsequence** $(f_{n_k})_{k\in\mathbb{N}}$ and a function $f \in C^0(X,Y)$ such that $f_{n_k} \to f$ uniformly.*

In the case the target space is a finite dimensional $\mathbb{R}$-vector space, say $\mathbb{R}^M$, we can give an easier characterization of compact sets of $C^0(X, \mathbb{R}^M)$. This is because Bolzano-Weierstrass gives an easier characterization of compact sets in $\mathbb{R}^M$

Definition 5. *Let (X, d_X) and (Y, d_Y) be two metric spaces, and let $F \subset C^0(X, Y)$. Let $F \subset C^0(X, Y)$. We say that the family F is equibounded if there exists $D < \infty$ such that*

$$\operatorname{diam}(f(X)) \leq D$$

for all $f \in F$.

Corrolary 2. *Let (X, d) be a **compact** metric space. Let $(f_n)_{n\in\mathbb{N}} \subset C^0(X, \mathbb{R}^M)$ be a sequence of **equibounded** and **equicontinuous** functions. Then, there exists a **subsequence** $(f_{n_k})_{k\in\mathbb{N}}$ and a function $f \in C^0(X, \mathbb{R}^M)$ such that $f_{n_k} \to f$ **uniformly**.*

Proof. Homework exercise. □

1.2 Separability for functions of a real variable (Theorems of Weierstrass)

$\mathbb{R}$ is separable, with a countable dense set being $\mathbb{Q}$. It is useful to be able to approximate functions with a countable set of functions of a certain class (e.g. polynomials!)

Weierstrass proved that algebraic polynomials $\mathbb{Q}[X]$ are dense in $C([0, 1])$

Proposition 1. *$f : [0, 1] \to \mathbb{R}$ of class C^0 and $\epsilon > 0$. Then*

$$\exists P : [0, 1] \to \mathbb{R} \text{ polynomial s.t. } |f - P|_{C^0} < \epsilon$$

Proof. (*Or, rather a sketch*) Note that by compactness of $[0, 1]$, f is uniformly continuous.

1. Approximate f with a spline (= a piecewise affine function). This can be done by considering a cover of $[0, 1]$ in $(x_i - \delta, x_i + \delta)$ where we chose δ s.t. for any x, y with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon/2$ (uniform continuity). Using this cover to create a partition of $[0, 1]$ by starting with $A_1 = x_1 \pm \delta$, then pick $A_i := [0, 1] \setminus \cup_{j=1}^{i-1} A_j$, then connect endpoints $(x_i, f(x, i))$ with secant lines, giving a uniformly ϵ-close spline.

2. Write the spline as a linear combination of absolute values (this can always be done, and is one of the homework exercises).

3. approximate $| \cdot |$ uniformly as a sequence of polynomials (duh.). But this can be done by approximating the square root function on $[0, 1]$: This you do using a uniformly monotonous sequence of polynomials, that converges pointwise to $t \mapsto \sqrt{t}$. Then use Dini's theorem to obtain uniform convergence.

□

2 Homework

Exercise 1 Prove that, given a $\epsilon > 0$ and a **continuous function** $f : [0, 1] \to \mathbb{R}$, there exists a **piecewise affine** function $g : [0, 1] \to R$such that $|f - g|_{C^0} < \epsilon$.

Proof Since $[0, 1]$ is compact and f is continuous, f is **uniformly continuous**. So for any $\epsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. So do this for $frac\epsilon 2 > 0$, and define the partition $\{[n\delta, (n + 1)\delta) : n = 0, 1, ..., \lceil \frac{1}{\delta} \rceil - 1\} \cup \{[(n + 1)\delta, 1]\}$. Call the interval starting at $k\delta$ the k-th interval I_k and define the right and left endpoints $lI = \inf I$, $rI = \sup I$ Then define $g_k : I_k \to \mathbb{R}$ through

$$g_k(x) = f(lI_k) + [f(rI_k) - f(lI_k)]x$$

In other words the linear interpolation of $(lI, f(lI))$ and $(rI, f(rI))$. Then set $g : [0, 1] \to \mathbb{R}$ as $g(x) := g_k(x)$ for $x \in I_k$ (unique because this is a partition of $[0, 1]$). g is piecewise affine and continuous: namely, g is the union of affine functions g_k, and these connect at the endpoints, because

$$g_k(lI_k) = f(lI_k) = f(rI_{k-1}) = g_{k-1}(rI_{k-1})$$

We now have, for $x \in I_k$, that

$$g(x) \in [g(lI_k), g(rI_k)] \cup [g(rI_k), g(lI_k)] \subset [\inf x \in I_k f(x), \sup_{x \in I_k} f(x)]$$

Since also $|\inf x \in I_k f(x) - \sup_{x \in I_k} f(x)| < \frac{\epsilon}{2}$ due to the δ-width of the interval, and $g(x)$ is between these values, we can conclude

$$\begin{aligned} |g(x) - f(x)| &\leq \max \left\{ |g(x) - \sup_{x \in I_k} f(x)|, |g(x) - \inf x \in I_k f(x) - \sup_{x \in I_k} f(x)| \right\} \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$

So we found, for an arbitrary $\epsilon > 0$, a piecewise affine g

Exercise 2 In this exercise we want to prove the first part of the Ascoli-Arzelà Theorem, in the case where the target space is $\mathbb{R}^M$. Let (X, d) be metric spaces, with X compact. Let $\mathcal{F} \subset C^0(X; RM)$ be a family of functions that is **compact** with respect to the **uniform convergence.** Prove that F is **equibounded** and **equicontinuous**. Namely, prove that:

(i) **Equibounded**. There exists $R > 0$ such that

$$|f(x)| \leq R$$

for all $f \in \mathcal{F}$, and all $x \in X$;

(ii) **Equicontinuous**. For each $\epsilon > 0$, it is possible to find $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in X$ with $d(x, y) < \delta$, and all $f \in \mathcal{F}$.

Proof

(i) Suppose not, then

$$\forall n \in \mathbb{N} : \exists x_n \in X : \exists f_n \in \mathcal{F} : |f_n(x_n)| \geq n$$

Since $\mathcal{F}$ is compact, we can extract a subsequence $f_{n_k}|$ such that $f_{n_k}| \to f \in \mathcal{F}$ uniformly, and we can extract a further sequence such that $x_{n_{k_l}} \to x \in X$. Since it still holds that $|f_{n_{k_l}}(x_{n_{k_l}}) \geq n_{k_l} \geq l$, we can assume w.l.o.g. that $(f_n)_n$ and $(x_n)_n$ are already convergent to f, x. Then, since $f_n \to f$ uniformly, it follows that we can take limits simultaneously:

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

So $(f_n(x_n))_n$ is a **bounded** sequence. But this is a contradiction with $|f_n(x_n)| \geq n, \ \forall n \in \mathbb{N}$

(ii) Suppose not. Then there is an $\epsilon > 0$ such that:

$$\forall n \in N : \exists x_n, y_n \in X, \ f_n \in \mathcal{F} : |y_n - x_n| < \frac{1}{n} \text{ and } |f_n(x_n) - f_n(y_n)| \geq \epsilon$$

We can w.l.o.g. take subsequences that converge since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq frac1k$, i.e. the statement still holds after having taken a subsequence ($\frac{1}{n_k} \leq \frac{1}{k}$ since the sublabelling $k \mapsto n_k$ is strictly increasing by definition). So assume that $f_n \xrightarrow{C^0} f \in \mathcal{F}$, $x_n \to x$ $y_n \to y$. Since $|x_n - y_n| < \frac{1}{n}$, we can conclude $y = x$, so they converge to the same limit. This implies, again, that we can take the limit simultaneously:

$$\lim_{n\to\infty} f_n(x_n) = f(x) \qquad \lim_{n\to\infty} f_n(y_n) = f(x)$$

Now, the difference $f_n(x_n) - f_n(y_n)$ then also converges as $n \to \infty$, namely to the difference $f(x) - f(x) = 0$. By continuity of the norm, $|f_n(x_n) - f_n(y_n)|$ also converges, and:

$$\lim_{n\to\infty} |f_n(x_n) - f_n(y_n)| = 0$$

Which contradicts $\forall n \in \mathbb{N} : |f_n(x_n) - f_n(y_n)| \geq \epsilon > 0$

Exercise 3 Let $X = C^1([a,b])$, and consider the C^1 norm on it, defined as

$$|f|_{C^1} := |f|_{C^0} + |f'|_{C^0}$$

, Prove that a family of functions $\mathcal{F} \subset X$is **compact** with respect to the C^1 norm if and only if

(i) $\mathcal{F}$ is **closed**

(ii) There exists $c \in [a, b]$ such that the set $\{f(c) : f \in \mathcal{F}\}$ is bounded in $\mathbb{R}$;

(iii) The family

$$\{f' : f \in F\} \subset C^0([a, b])$$

is **equibounded** and **equicontinuous**.

Proof

$\Longrightarrow$ Any compact set is closed, so if $\mathcal{F}$ is C^1-compact, then it is C^1-closed (i).

Moreover, Since $|\cdot|_{C^1} = |\cdot|_{C^0} + |\frac{d}{dx} \cdot |_{C^0}$, it follows that for a sequence to converge in C^1-norm, it has to be convergent in C^0 norm and its derivative should be convergent in C^0-norm (since the norms are nonnegative so they should go to 0 separately). This implies that $\mathcal{F}$ is C^0-compact, hence by using Ascoli-Arzelà it follows that $\forall x \in [a, b] : \{f(x) : f \in \mathcal{F}\}$ is compact, so in particular bounded, so there is certainly a $c \in [a, b]$ so that $\{f(c) : f \in \mathcal{F}\}$ is bounded (ii).

Finally, $\mathcal{F}' := \{f' : f \in \mathcal{F}\}$ is compact, since if $(g_n)_n$ is a sequence in $\mathcal{F}'$, we can consider a sequence $(f_n)_n$ in $\mathcal{F}$ such that $f_n' = g_n$, then extract a C^1-convergent subsequence to an $f \in \mathcal{F}$ and observe that by $f_n - f|_{C^1} \to 0$, we also have $|g_n - f'|_{C^0} \to 0$ and since $f' \in \mathcal{F}$, conclude that we found a convergent subsequence of g_n that converge in C^0-norm to a function in $\mathcal{F}'$. By Ascoli-Arzelà, we can conclude that $\mathcal{F}'$ must therefore be **equibounded** and **equicontinuous** (iii).

$\Longleftarrow$ Let (f_n) be a sequence in $\mathcal{F}$. Then $(f_n')_n$ is a sequence of equibounded and equicontinuous functions, therefore (by a corollary of Arzoli-Ascelà) we can extract a C^0-converging subsequence f_{n_k}', which we relabel immediately: assume $f_n' \xrightarrow{C^0} g \in \mathcal{F}$. Next, let c be as in (ii). Since $\mathcal{F}$ is C^1-closed, it is also C^0-closed, therefore $\{f(c) : f \in \mathcal{F}\}$ is closed, because otherwise we can find a sequence $(h_n)_n$ such that $(h_n(c))_n$ does not converge in $\mathcal{F}(c) = \{f(c) : f \in \mathcal{F}\}$, but then $(h_n)_n$ cannot C^0-converge in $\mathcal{F}$ since it does not converge pointwise. Therefore, $\mathcal{F}(c)$ is closed, and also is bounded. So it is compact by Bolzano-Weierstrass. This means we can extract a further subsequence, which we relabel again, such that $f_n(c) \to y = p(c)$ for some $p \in \mathcal{F}$. Using the fact that f_n is differentiable, and that its derivative is continuous, hence Riemann-integrable, write

$$f_n(x) = f_n(c) + \int_c^x f_n'(s) ds$$

Where the convention is that if $x < c$, then $\int_c^x := -\int_x^c$. Define $f \in C^1$

through $f(x) = y + \int_c^x g(s)ds$. Then,

$$\begin{aligned}
0 \leq |f_n - f|_{C^1} &= \left| f_n(c) - y + \int_c^x (f_n'(s) - g(s))ds \right|_{C^0} + |f_n' - g|_{C^0} \\
&\leq |f_n(c) - y| + \left| \int_c^x (f_n'(s) - g(s))ds \right|_{C^0} + |f_n' - g|_{C^0} \\
&\leq |f_n(c) - y| + \sup_{x \in [a,b]} \left| \int_c^x |f_n'(s) - g(s)| ds \right| + |f_n' - g|_{C^0} \\
&\leq |f_n(c) - y| + \sup_{x \in [a,b]} \left| \int_c^x |f_n' - g|_{C^0} ds \right| + |f_n' - g|_{C^0} \\
&= |f_n(c) - y| + \sup_{x \in [a,b]} |c - x| \cdot |f_n' - g|_{C^0} + |f_n' - g|_{C^0} \\
&\leq |f_n(c) - y| + |b - a| \cdot |f_n' - g|_{C^0} + |f_n' - g|_{C^0}
\end{aligned}$$

And since this is just a linear combination of two sequences that converge to 0 (by the extraction of appropriate subsequences), it follows that we can bound this from above by any $\epsilon > 0$ if we choose $n \geq N$ sufficiently large, so $f_n \xrightarrow{C^1} f$ as $n \to \infty$. All that is left is to show $f \in \mathcal{F}$, but this follows by the premise that $\mathcal{F}$ is C^1-closed and $f_n \xrightarrow{C^1} f$. Therefore $f \in \mathcal{F}$.