Analysis 2, Chapter 3

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1 Continuous Functions

By convention (X, d_X) and (Y, d_y) are metric spaces, with metrics d_X, d_Y, unless otherwise specified. In $\mathbb{R}^N$, we will denote the i-th standard basis vector as e_i.

1.1 Continuity in Metric Spaces and Topological Spaces

Definition 1. *$f : X \to Y$ is called*

(i) ***separately continuous*** *at $b \in X$ if $X = \mathbb{R}^N$, if $t \mapsto f(b + te_i)$ is continuous for every $i = 1, ..., N$.*

1. ***linearly continuous*** *at b if $X = \mathbb{R}^N$, and if $t \mapsto f(b + tv)$ is continuous for every vector $v \in \mathbb{R}^N$.*

2. ***topologically continuous*** *at $b \in X$, for general topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, if:*

$$\forall A \in \mathcal{T}_Y : f(b) \in A : \exists B \in \mathcal{T}_X : b \in B \wedge f(B) \subset A$$

3. ***sequentially continuous*** *at b, in general topological spaces$(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, if:*

$$\forall (x_n)_{n\in\mathbb{N}} \subset X : \lim_{n\to\infty} x_n = b : \lim_{n\to\infty} f(x_n) = f(b)$$

Proposition 1. *In metric spaces X, Y, the notion of sequential continuity and topological continuity at b are equivalent and are equivalent to:*

$$\forall \epsilon > 0 : \exists \delta > 0 : f(B_\delta(x)) \subset B_\epsilon(f(b))$$

Proposition 2. *Topological continuity on every point of X is equivalent to:*

$$\forall A \subset Y : A \text{ open} : f^{-1}(A) \text{ is open}$$

Remark 1. *In general topological spaces, continuity implies sequential continuity, but not the other way around.*

Proposition 3. *d_X is sequentially continuous:*

$$\forall (x_n)_n, (y_n)_n \subset X : x_n \to b, y_n \to \bar{y} \implies d_X(x_n, y_n) \to d(b, \bar{y})$$

Proposition 4. *Let $K \subset X$ compact and $f : X \to Y$ continuous: then $f(K) \subset Y$ is compact. This holds for general topological continuity.*

Proof. The simple proof of this is as follows: let $\{A_\alpha\}_{\alpha \in I}$ be an open cover of $f(K)$. Then $\{f^{-1}(A_\alpha)\}_{\alpha \in I}$ is an open (by topological continuity every pre-image is open) cover of K, so pick a finite subcover $\{f^{-1}(A_i)\}_{i=1}^N$ that covers K. Then

$$\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} f(f^{-1}(A_i)) = f(\bigcup_{i=1}^{N} f^{-1}(A_i)) \supset f(K)$$

So we have found a finite subcover for $f(K)$. □

Theorem 1. *Let $f : X \to \mathbb{R}$ be a continuous function and $K \subset X$ be compact. Then f assumes a maximum and minimum value on K, meaning $f(K) \subset \mathbb{R}$ has a maximum and minimum.*

Proposition 5. *Let $f : X \to Y$ continuous and $S_y = \{x \in X : f(x) = y\}$, for some $y \in Y$. Then S_y is closed.*

Moreover if $Y = \mathbb{R}$ with the Euclidean metric, then:

$$S_y^< = \{x \in X : f(x) < y\}, \quad S_y^> = \{x \in X : f(x) > y\}$$

are open.

Proposition 6. *Let $f : X \to Y$ be continuous at $x \in X$, and $g : Y \to Z$ continuous at $f(x) \in Y$. Then $g \circ f : X \to Z$ is continuous at x*

Proof. Again, a general topological result: let $A \subset Z$ be open and containing $g(f(x))$, then we know that there is an open subset B of Y such that $g(Y) \subset A$ and $f(x) \in B$. Now do the same for B to obtain an open set $\Omega \subset X$ containing x and $g(f(\Omega)) \subset g(A) \subset B$ □

Proposition 7. *If $f, g : X \to V$ are continuous, and V is a normed vector space. Then $\lambda f + g : X \to \mathbb{R}$ is also continuous.*

Proposition 8. *For $f, g : X \to \mathbb{R}$ continuous, the pointwise defined*

$$\max\{f, g\}, \quad \min\{f, g\} : X \to \mathbb{R}$$

are continuous. For a (possibly uncountable) collection $\{f_\alpha\}_{\alpha \in I}$, also the pointwise defined

$$\inf_{i \in I} f_i, \quad \sup_{\in I} f_i$$

are continuous (if they exist: we only define them if, for each $x \in X$, that $\{f_i(x) : i \in I\}$ is bounded), as well as $f \cdot g, \frac{f}{g}$ as long as the division is well-defined, meaning $g(X) \not\ni 0$.

Proof. Fix $x \in X$ and $\epsilon > 0$. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence converging to x. Let $f := \sup_\alpha f_\alpha$. We need to prove $f(x_n) \to f(x)$.

First, use the supremum property: let $\alpha \in I$ such that for this $x \in X$, we have:

$$f(x) - \epsilon < f_\alpha(x) \leq f(x) \text{ in } \mathbb{R}$$

By continuity of f_α, there is an $N \in \mathbb{N}$ with $\forall n \geq N : |f_\alpha(x_n) - f_\alpha(x)| < \epsilon$ For those n, we therefore have:

$$f(x) - 2 \cdot \epsilon < f(x_n) < f(x) + \epsilon$$

Which implies we have succesfully approximated $f(x_n)$ for $n \geq N$:

$$|f(x) - f(x_n)| < 2 \cdot \epsilon$$

The proof for the infinimum is almost the same, of course. □

Remark 2. *Let's look at an example for an* ***uncountable*** *index set I:*

$$f_y(x) := \frac{xy^2}{x^2 + y^2}, \quad y \in \mathbb{R}$$

Then

$$\sup_{y\in\mathbb{R}} f_y(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0 \end{cases}$$

It is continuous.

1.2 Uniform Continuity and Lipschitz Continuity

Definition 2. *A function $f : X \to Y$ is called* ***uniformly continous*** *on X if*

$$\forall \epsilon > 0 : \exists \delta > 0 : \forall x \in X : f(B_\delta(x)) \subset B_\epsilon(f(x))$$

Lemma 1. *Let $f : K \to Y$ be defined on a* ***compact*** *metric space K. Then f is* ***continuous*** *if and only if it is* ***uniformly continuous***

Proof. (Sketch) Fix, for each $x \in K$ and $\epsilon > 0$ a $\delta_x > 0$ such that $\forall y \in B_{\delta_x}(x)$ we have $d_Y(f(x), f(y)) < \epsilon$. Then cover K **not** with δ_x-balls but with $\frac{1}{2}\delta_x$-balls:

$$\{B_{\frac{1}{2}\delta_x}(x) : x \in K\}$$

Is a cover for K. Use this to extract a subcover

$$\{B_{\frac{1}{2}\delta_i}(x_i)\}_{i=1}^N$$

We can take the minimum of a **finite set**. Assume w.l.o.g. that $\delta_1 = \min_{i=1,\ldots,N} \delta_i$. Conclude that **if** $d_X(y, z) < \frac{1}{2}\delta_1$, and by coverage $x \in B_{\frac{1}{2}\delta_i}(x_i)$, for some i, **then**:

$$d_X(z, x_i) \leq d_X(z, x) + d_X(x_i, x) < \frac{1}{2}\delta_1 + \frac{1}{2}\delta_i < \delta_i$$

Therefore $y, z \in B_{\delta_i}(x_i)$ for some i by the triangle inequality, and we conclude $d_Y(f(z), f(y)) < \epsilon$. We find $\frac{1}{2}\delta_1$ to be the appropriate delta. □

An equivalent definition of uniform continuity, is that f *admits a modulus of continuity:*

Definition 3. *A* ***modulus of continuity*** *for a function* $f : X \to Y$ *is a function* $\omega : [0, \infty) \to [0, \infty)$ *that:*

- *is* ***non-decreasing.*** $\forall st : t \geq s \implies \omega(t) \geq \omega(s)$
- *is* ***right-continuous*** *at* 0 *with* $\lim_{t \downarrow 0} \omega(t) = \omega(0) = 0$
- *satisfies*
$$d_Y(f(x), f(\tilde{x})) \leq \omega(d_X(x, \tilde{x}))$$

Interesting (somewhat esoteric) fact: By its properties, ω is itself also uniformly continuous.

Proposition 9. *If* $(f_n)_n$ *is a sequence of functions that all admit* ***the same modulus*** ω*, and such that* $\forall x \in X : f_n(x) \to f(x)$ *for some* $f : X \to Y$ *(**pointwise convergence**). Then* ω *also is a modulus of continuity for* f

Proof. This follows from the fact that for fixed $x, y \in X$:

$$\forall i \in \mathbb{N} : d_Y(f_i(x), f_i(y)) \leq \omega(d_X(x, y))$$

And we simply take the limit and preserve the inequality. □

Proposition 10. *If* $(f_i)_{i \in I}$ *is any (possibly uncountable) collection of functions that all admit the* ***same modulus*** ω*. Then*

$$\inf_{i \in I} f_i \qquad \sup_{i \in I} f_i$$

also admit that modulus.

Proof. Again, we have for fixed $x, y \in X$:

$$\forall i \in I : d_Y(f_i(x), f_i(y)) \leq \omega(d_X(x, y))$$

Therefore, we can take the supremum over I (which may be uncountable now):

$$d_Y\left(\sup_{i \in I} f_i(x), \sup_{i \in I} f_i(y)\right) \leq \sup_{i \in I} d_Y(f_i(x), f_i(y)) \leq \omega(d_X(x, y))$$

Showing that ω is a modulus of continuity for $\sup_{i \in I} f_i$. □

Definition 4. $f : X \to Y$ *is called* ***Lipschitz continuous*** *if it admits a* ***linear modulus of continuity*** $\omega \in \mathcal{L}(\mathbb{R}, \mathbb{R})$*. The constant* $L > 0$ *such that* $\omega(\cdot) = L\cdot$*, is called its* ***Lipschitz constant***

Since a linear modulus of continuity is also a modulus of continuity, we can conclude:

Proposition 11. *A collection $\{f_i\}_{i\in I}$ of Lipschitz continuous functions sharing **the same Lipschitz constant** is closed under **pointwise convergence, and taking suprema/infinima.***

Definition 5. *More generally, a function $f : X \to Y$ is called α**-Hölder continuous** if it has a modulus of continuity that is a power relation $\omega(t) = Lt^\alpha$, where $\alpha > 0$.*

If $\alpha = 0$, then the function is bounded. If $\alpha > 1$ and $X = \mathbb{R}^N$, we can show that f is constant.

1.3 Pointwise and Uniform convergence

Definition 6. *$(f_n)_n$ a sequence of functions $f_n : X \to Y$ is said to **converge pointwise** to $f : X \to Y$ if*

$$\forall x \in X : (f_n(x))_n \text{ converges to } f(x) \text{ in } X$$

Definition 7. *$(f_n)_n$ a sequence of functions $f_n : X \to Y$ is said to **converge uniformly** to $f : X \to Y$ if the N_ϵ can be picked independently of x:*

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \forall x \in X : d_y(f_n(x), f(x)) < \epsilon$$

We can equivalently say that $f_n \to f$ with respect to the the **uniform metric**

$$d_\infty(f, g) := \sup_{x\in X} d_Y(f(x), g(x))$$

But this is only well-defined once we show that d_∞ is a metric, i.e. we require $f(X)$ to be bounded for all f in a certain **function space**. See Chapter 4.

We also call d_∞ the **supremum metric**, **uniform norm** or even *supremum norm*, depending on whether the space Y is a normed space or not.

Proposition 12. *Uniform convergence preserves continuity: i.e. if $\forall n \in \mathbb{N}$: f_n is **continuous** at $x \in X$, then f is also **continuous** at $x \in X$ if $f_n \to f$ **uniformly**.*

Proof. We use the triangle inequality:

$$d_Y(f(x), f(y)) \leq d_Y(f_n(x), f(x)) + d_Y(f_n(y), f(y)) + d_Y(f_n(x), f_n(y))$$

Which we can bound by ϵ if $n \geq N_{\frac{1}{3}\epsilon}$ and $d_X(x, y) < \delta_{\frac{1}{3}\epsilon}$. We have to use **uniform contuity** since we have to bound $d_Y(f_n(x), f(x))$ and $d_Y(f_n(y), f(y))$ simultaneously for arbitrary $y \in B_{\delta_{\frac{1}{3}\epsilon}}(x)$. □

Proposition 13. *Using almost exactly the same proof, one can show that **uniform convergence preserves uniform continuity.***

A more interesting fact:

Proposition 14. *Let $(f_n)_n$ a sequence of functions $f_n : X \to Y$ with uniform convergence to a continuous function $f : X \to Y$. Note that we don't assume any f_n to be continuous. Now, if $x_n \to x$ for some sequence $(x_n)_n \subset X$, we get:*

$$f_n(x_n) \to f(x)$$

Proof. Not actually very surprising: we can pick N large enough that $d_X(x_n, x) \leq \delta_\epsilon$ where δ_ϵ is s.t. $f(B_{\delta_\epsilon}(x)) \subset B_\epsilon(f(x))$. Next, pick N large enough so that $d_\infty(f_n, f) < \epsilon$. Then use some triangle inequality magic:

$$d_Y(f_n(x_n), f(x)) \leq d_Y(f(x_n), f(x)) + d_Y(f_n(x_n), f(x_n)) < 2\epsilon$$

□

The reason that this proof fails if $f_n \to f$ only **pointwise**, is that we cannot bound the term $d_Y(f_n(x_n), f(x_n))$, because we don't know how large N should be, because we cannot pick it uniformly: it may depend on x_n. And that is also the strategy if you wanted to find a counterexample.

The possibility to take the limit of an argument sequence simultaneously with the limit of the function sequence, characterizes uniform convergence of the function sequence, provided X is compact:

Proposition 15. *Let $(f_n)_n$ be a sequence of functions $X \to Y$, where X, Y are metric spaces. If:*

*(i) X is **compact**.*

*(i) There is a function $f : X \to Y$ such that: for all sequences $(x_n)_n \subset X$ that are **convergent**, with $x_n rab$, we have that $(f_n(x_n))_n$ is convergent and **converges to** $f(b)$*

Then $f_n \to f$ uniformly.

Proof. **Homework, Exercise 3.** □

We repeat two theorems regarding uniformly continuous functions and Riemann integrability from Analysis 1:

Proposition 16. ***Uniformly continuous** real functions f on **compact** sets K, i.e. $f : K \to \mathbb{R}$ are **Riemann integrable,** and thus also are continuous functions on compact sets.*

Proposition 17. *For $(f_n)_n$ a sequence of **uniformly continuous** scalar functions on a compact set $K \subset \mathbb{R}$, converging **uniformly** to f, then f is uniformly continuous, hence **Riemann integrable** and:*

$$\lim_{n\to\infty} \int_K f_n(x)dx = \int_K f(x)dx$$

Note that this writes as:

$$\lim_{n\to\infty} \int_K f_n(x)dx = \int_K \left(\lim_{n\to\infty} f(x)\right) dx$$

As a consequence (using the *fundamental theorem of calculus*):

Proposition 18. *Let $(f_n)_n$ be a sequence of* ***scalar, differentiable functions*** *$f_n : I \to \mathbb{R}$, defined on an* ***interval*** *$I \subset \mathbb{R}$, which* ***converges pointwise*** *to a function $f : I \to \mathbb{R}$ and whose derivatives $(f'_n)_n$* ***converge uniformly*** *to a $g : I \to \mathbb{R}$. Then f is differentiable and $f' = g$.*

1.4 Dini's theorem

Definition 8. *A sequence $(f_n)_{n\in\mathbb{N}}$ of scalar functions $f_n : X \to \mathbb{R}$ on a set X. Is said to be* ***increasing*** *if*

$$\forall x \in X : \forall n \in \mathbb{N} : f_n(x) \leq f_{n+1}(x)$$

And we say $(f_n)_{n\in\mathbb{N}}$ is strictly increasing if the inequality is strict. We can define decreasing and strictly decreasing likewise. Further, if $(f_n)_n$ either decreasing or increasing, we call the sequence ***monotone****.*

Theorem 2. ***Dini's Theorem***

Let (X, d) be a metric space, and X ***compact****. Let $(f_n)_{n\in\mathbb{N}}$ be a* ***monotone sequence of continuous functions*** *$f_n : X \to \mathbb{R}$, that converges* ***pointwise*** *to a function $f : X \to \mathbb{R}$. Then the convergence is* ***uniform****.*

Proof. W.l.o.g., assume monotone means increasing. For decreasing sequences, consider $-f_n$ and $-f$. Then, we define $g_n : f - f_n$ and show $g_n \to 0$ uniformly.

Let $E_n \subset X$ be defined as:

$$E_n := \{x \in X : g_n(x) < \epsilon\}$$

By continuity of g_n, we know E_n is open, and by pointwise convergence, we know

$$\forall x \in X : \exists N \in \mathbb{N} : \forall n \geq N : x \in E_n$$

And by monotonicity of the f_n, if $x \in E_n$, then $f_{n+1}(x) \leq f_n(x) < \epsilon$, hence $E_n \subset E_{n+1}$. Therefore, E_n is an **ascending** sequence: $E_n \subset E_{n+1}$, and moreover $E_n \uparrow X$, meaning

$$\bigcup_{n\in\mathbb{N}} E_n = X$$

An open cover of X has a finite subcover, so we can find $N_1, ...N_k$ with

$$\bigcup_{i=1}^{k} E_{N_i} = X$$

Setting $N := N_k$, we see $X = E_N$, meaning $\forall n \geq N : \forall x \in X : g_n(x) < \epsilon$ We conclude that N is uniform in $x \in X$, and this can be done for each $\epsilon > 0$.

□

Remark 3. *Why can we not do this for arbitrary compact X, continuous f_n and continuous f, setting $g_n(x) := |f_n(x) - f(x)|$? In other words, can we drop monotonicity?*

To see the problem, consider the following counterexample: $_n, f : [0,1] \to \mathbb{R}$, where $f = 0$ and

$$f_n(x) := \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} < x \leq 1 \end{cases}$$

Here, we have the following level sets:

$$\begin{aligned} E_n &= \left\{ x : x \leq \frac{1}{n} \text{ and } nx \leq \epsilon, \text{ or } x > \frac{1}{n} \text{ and } 2 - nx < \epsilon, \text{ or } x > \frac{2}{n} \right\} \\ &= \left[0, \frac{\max(1,\epsilon)}{n}\right) \cup \left(\frac{\min(2-\epsilon, 1)}{n}, 1\right] \end{aligned}$$

That is, for $\epsilon = \frac{1}{2}$, we get:

$$E_n = \left[0, \frac{1}{2n}\right) \cup \left(\frac{3}{2n}, 1\right]$$

Or, maybe more clarifying, is the case $\epsilon = 1$:

$$E_n = [0,1] \setminus \left\{\frac{1}{n}\right\}$$

*This shows that, indeed, for each $x \in [0,1]$ there is an N such that $x \in E_n$, $\forall n \geq N$ (indeed this holds for all x except those of the form $\frac{1}{k}$, $k \in \mathbb{N}$, and for such x we have $x \in E_n$, $\forall n \geq k+1$). But we need **monotonicity** of $(g_n)_{n\in\mathbb{N}} = (x \mapsto |f_n(x) - f(x)|)_{n\in\mathbb{N}}$ to conclude that $E_n \subset E_{n+1}$, which is not the case here. Therefore, we can indeed say, for any $\epsilon > 0$,*

$$\bigcup_{n\in\mathbb{N}} E_n = [0,1]$$

*And surely we can for any $\epsilon > 0$, extract a finite subcover $E_{n_1}, ..., E_{n_N}$. But since $(E_n)_{n\in\mathbb{N}}$ is **not ascending**, we cannot conclude that $E_{n_N} = X$, and in this case, we can **never** find an N such that $E_N = X$. This shows that monotonicity is a strictly necessary premise.*

2 Homework

Exercise 1 Let (X, d) be a metric space, and let $E \subset X$ be a set. Consider the function $f : X \to \mathbb{R}$ defined as

$$f(x) := d(x, E)$$

where the distance of the point $x \in X$ to the set E is defined as

$$d(x, E) := \inf\{d(x, y) : y \in E\}.$$

Prove that f is continuous.

Proof If E is nonempty, then f is, firstly, well-defined since $\{d(x, y) : y \in E\}$ is bounded from below by 0, by definiteness of the metric d.

Now, let $(x_n)_{n\in\mathbb{N}}$ be any sequence in X such that $\lim_{n\to\infty} x_n$ exists, say it is $x \in X$. We need to show that $\lim_{n\to\infty} f(x_n) = f(x)$

Note that for any $z \in E$, it holds by using the triangle inequality, that:

$$\begin{aligned} d(x_n, z) &\leq d(x_n, x) + d(x, z) \\ d(x, z) &\leq d(x_n, x) + d(x_n, z) \end{aligned}$$

In particular, taking the infinimum on both sized over all $z \in E$ (which can be done since $d(y, z)$ is bounded from below by 0):

$$\begin{aligned} d(x_n, E) &\leq d(x_n, x) + d(x, E) &&\implies d(x_n, E) - d(x, E) \leq d(x_n, x) \\ d(x, E) &\leq d(x_n, x) + d(x_n, E) &&\implies d(x, E) - d(x_n, E) \leq d(x_n, x) \end{aligned}$$

This implies the inequality

$$0 \leq |d(x_n, E) - d(x, E)| \leq d(x_n, x), \quad \text{or: } 0 \leq |f(x) - f(x_n)| \leq d(x_n, x)$$

Now, since $x_n \to x$, this means precisely $d(x_n, x) \to 0$ as $n \to \infty$. By the above and the **squeeze theorem**, it follows that $|f(x) - f(x_n)|$ converges to 0. This, by definition, means $f(x_n) \to f(x)$, so f is sequentially continuous (hence continuous).

Exercise 2 Let $F : C^1_p([0, 1]) \to C^1_p([0, 1])$ be the identity map, where $C^1_p([0, 1])$ is the space of functions that are piecewise C^1. Namely, $f \in C^1_p([0, 1])$ if and only if $f \in C^0([0, 1])$, and there are finitely many points $0 = x_0 < x_1 < x_2 < ... < x_k = 1 \in [0, 1]$ such that

$$f \in C^1((x_i, x_{i+1})) \text{ for all } i = 0, ..., k - 1$$

Endow the domain of F with the C^0 norm and the codomain of F with the C^1 norm. Prove that the identity is not continuous.

Proof This means that we need to find a sequence of function $f_n \in C^1_p([0, 1])$ such that $f_n \to f \in C^1_p([0, 1])$ in C^0-norm but not in C^1-norm. Since $|f - h|_{C^1} = |f - h|_{C^0} + |f' - h'|_{C^0}$, this means that $|f' - f'_n|_{C^0} > \epsilon$ for some $\epsilon > 0$. For example,

$$f_n(x) := \frac{\sin nx}{n}$$

Which has uniform limit $0 \in C^1_p([0, 1])$ (the zero function, which is $C^\infty([0, 1])$, because $|f_n - 0|_{C^0} = \frac{1}{n} \sup_{x\in[0,1]} |\sin nx| = \frac{1}{n} \to 0$ as $n \to \infty$. But for its derivative, $f_n'(x) = \cos(nx)$, we have $|f_n' - 0|_{C^0} = \sup_{x\in[0,1]} |\cos nx| = 1$, which does not converge to 0 as $n \to \infty$.

Hence,

$$\lim_{n\to\infty} |f_n - 0|_{C^0} = 0, \quad \text{but} \lim_{n\to\infty} |F(f) - F(0)|_{C^1} = 1 > 0$$

So that $f_n \xrightarrow{C^0}$ yet $F(f_n) \overset{C^1}{\not\longrightarrow} F(0)$, showing that F is not continuous everywhere on $C_p^1([0,1])$.

Exercise 3 Let $(f_n)_{n\in N}$ be a sequence of functions from two metric spaces (X, d_X) and (Y, d_Y), and let $f : X \to Y$ be a continuous function. Then:

(i) Assume that $(f_n)_{n\in N}$ converges uniformly to $f : X \to Y$. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every $x \in X$ and every sequence of points $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$;

(ii) Find a counterexample in the case the convergence is only pointwise, but not uniform;

(iii) Assume that X is compact. Assume that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every $x \in X$ and every sequence of points $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$. Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Proof

(i) Let $\epsilon > 0$. We can, by uniform convergence of $(f_n)_{n\in\mathbb{N}}$, find an $N_1 \in \mathbb{N}$ such that for all $n \geq N$:

$$\forall z \in X : d_Y(f_n(z), f(z)) < \epsilon$$

In particular, for all $n \geq N$

$$d_Y(f_n(x_n), f(x_n)) < \epsilon$$

By continuity of f and the fact that $(x_n)_{n\in\mathbb{N}}$ converges to x, it follows that $f(x_n)$ converges to $f(x)$, so that we can also find an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$:

$$d_Y(f(x_n), f(x)) < \epsilon$$

Combining this information, for $N = \max\{N_1, N_2\}$, for all $n \geq N$, we have:

$$d_Y(f_n(x_n), f(x)) \leq d_Y(f_n(x_n), f(x_n)) + d_Y(f(x_n), f(x)) < 2\epsilon$$

Where the inequality is just the triangle inequality for the metric d_Y. This, by definition, means that $d_Y(f_n(x_n), f(x)) \to 0$ as $n \to \infty$, or $f_n(x_n) \to f(x)$ as $n \to \infty$.

(ii) Define $f_n : [0,1] \to \mathbb{R}$ as:

$$f_n(x) := \begin{cases} x^n & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

We see that f_n is discontinuous, but it converges to 0 pointwise, which is continuous. The convergence is not uniformly since for any $n \in \mathbb{N}$, $|f_n - 0|_{C^0} = \sup_{x\in[0,1)} |x^n| = 1$ which does not converge to 0. Now consider the sequence $(1 - \frac{1}{n})_{n\in\mathbb{N}}$. This sequence converges in $[0,1]$, namely to 1. Yet,

$$\lim_{n\to\infty} f_n(x_n) = (1 - \frac{1}{n})^n = e^{-1}, \quad \text{while } 0(1) = 1$$

So this is a counterexample.

(iii) An argument by contradiction: Suppose that $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly to f. The negation of the statement "converges uniformly" is:

$$\exists \epsilon > 0 : \forall N \in \mathbb{N} \exists n \geq n_N : \exists x_N \in X : d_Y(f_{n_N}(x_N), f(x)) \geq \epsilon$$

So pick this ϵ and let $(f_{n_k})_{k\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$ be defined in this way. Since $(x_n)_{n\in\mathbb{N}} \subset X$ and X is compact, we can assume without loss of generality that $(x_n)_{n\in\mathbb{N}}$ converges to a limit $a \in X$, since else we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to a limit $a \in X$, that also satisfies, for $(x_{n_{l_k}})_{k\in\mathbb{N}}$ the further subsequence of f, $d_Y(f_{n_{l_k}}(x_{n_k}), f(x)) \geq \epsilon$.

Next, the triangle inequality gives:

$$d_Y(f_n(x_n), f(x)) + d_Y(f(x_n), f(x)) \geq d_Y(f_n(x_n), f(x_n)) \geq \epsilon$$

Taking the limit on both sides, which we can do since $\lim_{n\to\infty} f_n(x_n) = f(x)$ and f is continuous, so $\lim_{n\to\infty} f(x_n) = f(x)$, we get:

$$0 + 0 \geq \epsilon > 0, \text{ a contradiction.}$$