Analysis 2, Chapter 2

Matthijs Muis

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1 Compactness and Banach's Fixed Point Theorem

1.1 Compactness in Metric Spaces

Definition 1. *Let (X, d) be a metric space. Then $A \subset X$ is called* ***sequentially compact*** *if for all sequences $(x_n)_{n\in\mathbb{N}} \subset A$, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that is convergent to a $a \in A$.*

Remark 1. *In particular, a sequentially compact set A is itself a complete metric space.*

Definition 2. *If (X, d) is a metric space, a set $E \subset X$ is called* ***bounded*** *if:*

$$\text{diam}(E) = \sup_{x,y\in E} d(x, y)$$

exists (and is $< \infty$, if we define the supremum of an unbounded set as ∞)

Lemma 1. *A sequentially compact set in a metric space is bounded.*

It is even **totally bounded**, which is stronger than being bounded, but we will define that later.

Definition 3. *A* ***sequentially closed set*** *is a set $C \subset X$ is a subset such that:*

$$\forall (c_n)_{n\in\mathbb{N}} \subset C : \lim_{n\to\infty} c_n = a \in X \implies a \in C$$

This is equivalent to: a sequentially closed set is a set C that equals its sequential closure $C = \overline{C}^d$

Remark 2. *In fact, we could have even defined $\overline{Y}^d$ as being the smallest sequentially closed set (that is, a minim**um** set with respect to $\subset$) that contains Y.*

Lemma 2. *A sequentially compact set in a metric space is sequentially closed.*

Definition 4. *A set $D \subset X$ where $(X, \mathcal{T})$ is a topological space is called* ***topologically closed*** *if its complement in X, $D^c = X\backslash C$, is open, i.e. $D^c \in \mathcal{T}$.*

Definition 5. *The **topological closure** $\overline{S}^{\mathcal{T}}$ of a set $S \subset X$, where (X, T) is a topological space, is the mini**mum** closed set (with respect to $\subset$) that contains S. This is well-defined, since if $S \subset X \setminus \Omega_i$ for $i \in I$, $\Omega_i \in T$, then $\bigcap_{i \in I} \Omega_i \in T$ hence $S \subset \cap_{i \in I} X \setminus \Omega_i$, so if there are two distinct **minimal** sets, their intersection is minimal too, and therefore, a minimum can always be found.*

Proposition 1. *We have the following interplay between $\overline{S}^{\mathcal{T}}$ and $\overline{S}^{d}$:*

1. *In a metric space, where we consider the topology generated by open balls $B_r(x)$, $r > 0$, $x \in X$, a set is topologically closed if and only if it is sequentially closed.*
2. *In a topological space, a set A is topologically closed if and only if A equals its topological closure.*

*This is why, in metric spaces, we can speak of **closures** and **closed sets** without specifying whether we mean this in the topological or sequential sense.*

Definition 6. *In a metric space (X, d), a set $A \subset X$ is called **dense** in X if all $x \in X$ are the limit of some sequence $(a_n)_{n=1}^{\infty} \subset A$. Equivalently,*

$$\forall x \in X : \forall \epsilon > 0 : \exists a \in A : d(x, a) < \epsilon$$

Or equivalently, $\overline{A}^{d} \supset X$

We can regard subsets $S \subset X$ as their own metric space with the restricted metric $d|_S$. In this way, we define what it means for a set D to be dense in a set S: this simply means that D is dense in the metric space (S, d).

The classic example is $\mathbb{Q} \subset \mathbb{R}$, the **Archimedean property**. By repeatedly (N times) picking a subsequence along every dimension of the space $\mathbb{R}^N$, we can extend this to $\mathbb{Q}^N \subset \mathbb{R}^N$ being dense with respect to the Euclidean metric. This is equivalent to the version of the Bolzano-Weierstrass theorem, which says every bounded sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N$ has a converging subsequence

Because $\mathbb{Q}$ is countable, we call $\mathbb{R}$ *separable*:

Definition 7. *A subset $S \subset X$ of a metric space is **separable** if there is a countable $D \subset S$ such that D is dense in S, i.e. if there is a $\{s_i\}_{i \in \mathbb{N}}$ that is dense in S. Equivalently, this reads that there is a countable set $\{s_i\}_{i \in \mathbb{N}} \subset S$ so that, for any $\epsilon > 0$, the set*

$$\{B_\epsilon(s_i)\}_{i \in \mathbb{N}}$$

covers S.

Proposition 2. *A sequentially compact set K in a metric space X is separable. This means that there is a countable set $\{k_i\}_{i \in \mathbb{N}}$ so that, for any $\epsilon > 0$,*

$$\{B_\epsilon(k_i)\}_{i \in \mathbb{N}}$$

covers K.

Remark 3. *The above proposition is a different statement than* ***total boundedness****:*

- ***separability*** *means:* $\exists S \subset K$, *countable* : $\forall \epsilon > 0 : \bigcup_{s \in S} B_\epsilon(s) \supset K$
- ***total boundedness*** *means:* $\forall \epsilon > 0 : \exists F_\epsilon \subset K$, *finite* : $\bigcup_{s \in F_\epsilon} B_\epsilon(s) \supset K$

The second statement implies the latter, for we can let $S = \bigcup_{n=1}^{\infty} F_{\frac{1}{n}}$, *which is a countable union of countable sets, hence countable.*

Definition 8. *A set* $S \subset X$ *is* ***totally bounded*** *if for any* $\epsilon > 0$, *there is a finite cover of* S *in* ϵ*-balls*

Total boundedness is *stronger* than boundedness: consider $(\mathbb{R}, d)$ with $d(x, y) = \min\{|x-y|, 1\}$. Then $\mathbb{R}$ is bounded, with $\text{diam}(\mathbb{R}) = 1$, but not totally bounded, since we cannot cover $\mathbb{R}$ with $\frac{1}{2}$-balls.

Proposition 3. ***Total boundedness*** *of* S *implies* ***separability****, as discussed in the preceding proposition.*

1.2 Compactness in Topological Spaces

Definition 9. *For a set* S, *a collection of sets* $\mathcal{C}$ *is said to* ***cover*** S *if*

$$S \subset \bigcup \mathcal{C}$$

Definition 10. *In a topological space* $(X, \mathcal{T})$, *a set* $K \subset X$ *is called* ***topologically compact*** *if for any cover of* K *in open sets, there is a* ***finite subcover****:*

$$\forall S \subset \mathcal{T} : \bigcup S \supset K : \exists S' \subset S \text{ finite } : \bigcup S' \supset K$$

Proposition 4. *In a metric space* (X, d), *a set* $K \subset X$ *is* ***topologically compact*** *with respect to the topology generated by open balls, if and only if it is* ***sequentially compact.***

Remark 4. *In a general topological spaces, sequential compactness may be strictly stronger than compactness. Note that sequential compactness is defined in a topological space, since convergence is defined in a topological space.*

Example: $(X, \mathcal{T})$ *where*

$$X = (0, 2) \text{ and } \mathcal{T} = \{\emptyset, (0, 1), (1, 0), X \backslash \{1\}, X\}$$

Then $(1, 2)$ *is compact since* $\mathcal{T}$ *is finite so every cover of* $(1, 2)$ *is necessarily finite. But it is not complete, so cannot be sequentially compact. Note that sequential compactness should be read in the topological sense, here.*

Theorem 1. *A* ***compact set*** *in a metric space is* ***complete****,* ***bounded****,* ***totally bounded****,* ***closed****,* ***separable****.*

But what conditions are sufficient to conclude compactness?

1.3 Characterizations of compactness

In a finite-dimensional real vector space, the Bolzano-Weierstrass Theorem from Real Analysis gives a characterization of compact sets:

Theorem 2. *(Bolzano-Weierstrass) $k \subset \mathbb{R}$ is* ***compact*** *with respect to the Euclidean norm if and only if it is* ***closed*** *and* ***bounded****. By extracting converging subsequences along every dimension, we can extend this to any finite dimensional real vector space $\mathbb{R}^N$ endowed with the Euclidean norm.*

Proof. The proof of Bolzano-Weierstrass essentially relies on the supremum property for bounded sets and the fact that we can always extract a monotone subsequence from an $\mathbb{R}$-sequence $(y_n)_n$ by considering for every $n \in \mathbb{N}$ the sets of indices:

$$H_n^{<} = \{i > n : y_i < y_n\}$$
$$H_n^{\geq} = \{i > n : y_i \geq y_n\}$$

At least one of these sets is infinite. And if we do this for every $n \in \mathbb{N}$, then either one of the two following is infinite:

$$\{n \in \mathbb{N} : H_n^{<} \text{ is infinite}\}$$
$$\{n \in \mathbb{N} : H_n^{\geq} \text{ is infinite}\}$$

So one of the two is gives an appropriate infinite subsequence of $(y_n)_n$, and this is by construction a monotone one. Note that by considering $H_n^{<}, H_n^{<}, H_n^{=}$, we can even extract strict monotone or constant subsequences (this is not required for the proof). And monotone bounded sequences converge, that is the final step, which relies on the supremum/infinimum property. □

Proposition 5. *In a* ***complete*** *metric space (X, d), $K \subset X$ is compact if and only if it is* ***totally bounded*** *and* ***closed****.*

Remark 5. *Even in a non-complete metric space, a compact set is totally bounded.*

But a closed, totally bounded set may fail to be compact in an incomplete metric space, for example $B = (0, 1)$ in the metric space $((0, 1), |\cdot - \cdot|)$: it is closed and totally bounded, yet not complete, so it cannot be compact.

1.4 Banach's fixed point theorem

Somewhat unrelated at first glance, is the fixed point theorem of Banach. Note that we use some terminology that will only be properly introduced in chapter 3.

Definition 11. *(X,d) a metric space. A map $f : X \to X$ is called a **contraction** if*

$$\exists \alpha \in [0,1) : \omega(t) = \alpha t \text{ is a modulus of continuity for } f$$

In other words:

$$\exists \alpha \in [0,1) : \forall x, y \in X : d(f(x), f(y)) \leq \alpha d(x,y)$$

Proposition 6. *Let (X,d) a **complete** metric space. If $f : X \to X$ is a contraction with lipschitz constant α, then f has a **fixed point** $x^* \in X$, which is a x^* such that $f(x^*) = x^*$. Moreover, the following approximation holds for any $x_0 \in X$:*

$$d(f^n(x_0), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, f(x_0))$$

From this approximation, it follows that x^* is unique. For if y^* is another fixed point, we can consider the constant sequence $(y^*)_{n\in\mathbb{N}} = (f^n(y^*))_{n\in\mathbb{N}}$, which by the approximation, must converge to x^*.

As a corrolary, one can prove a version of the **Picard-Lindelöf theorem**:

Proposition 7. *Let $f : [-1,1] \times K \to \mathbb{R}^N$, where $K \subset \mathbb{R}^N$ is **compact** and **connected**. Assume f is continuous and*

$$\forall s \in [-1,1] : p \mapsto f(s,p) \text{ is Lipschitz}$$

Then, there is an $\epsilon > 0$ s.t. the initial value problem (for $y_0 \in K$)

$$\begin{cases} y'(t) & = f(t, y(t)) \\ y(0) & = y_0 \end{cases}$$

Has a unique solution $y : (-\epsilon, \epsilon) \to K$.

The proof for $N = 1$ and $K = \mathbb{R}$ (not actually compact, so somewhat of a different theorem) is homework.

2 Homework

Exercise 1 Let (X,d) be a metric space, and let $K \subset X$ be a compact set. Without using the equivalence between compact and sequentially compact sets, prove that:

(i) K is bounded;

(ii) K is closed

Proof

1. Suppose that K is not bounded; consider an $x \in X$ (since K is unbounded, X is nonempty), and the family of balls:

$$\{B_r(x) \mid r > 0\}$$

This family covers X since $d(x,y) \in [0,\infty)$ for all $y \in X$. Therefore, it covers K. But since K is unbounded, it is not *contained* in a $B_r(x)$, for *any* $r > 0$. Therefore, it is not possible to extract a finite subcover $\{B_{r_i}(x)\}_{i=1,\dots,N}$, since then w.l.o.g. (up to reordering the indices) assume $r_1 = \max_{i=1,\dots,N} r_i$, then it follows that $B_{r_i}(x) \subset B_{r_1}(x)$, $\forall i = 1, ..., N$, so we can actually cover K with $\{B_{r_1}(x)\}$, meaning $K \subset B_{r_1}(x)$, a contradiction with unboundedness of K.

2. It is sufficient to show that K^c is open. Let $x \in K^c$ (if $K^c = X \backslash K = \emptyset$, then we are done since $X = K$, and K is always closed in itself). Define for all $y \in K$, $r_y = \frac{d(x,y)}{2}$. Then the following is obviously a cover for K:

$$\left\{B_{r_y}(y) \mid y \in K\right\}$$

Since all $y \in K$ are at least contained in their own ball. By compactness of K, extract a finite subcover $\{B_{r_i}(y_i)\}_{i=1,\dots,N}$. Here we denote r_{y_i} with r_i simply. Assume w.l.o.g. (up to reordering the indices) that $r_1 = \min_{i=1,\dots,N}$. A minimum over a finite set always exist, so here we use the fact that the cover is finite. Then, $B_{r_1}(x) \cap B_{r_i}(y_i) = \emptyset$ for any $i = 1, ..., N$, since if there is a z in this intersection, by the triangle inequality $2r_i = d(x, y_i) \le d(x,z) + d(z, y_i) < r_1 + r_i \le 2r_i$, contradiction.

This implies

$$\left[\bigcup i = 1^N B_{r_i}(y_i)\right] \cap B_{r_1}(x) = \emptyset$$

But since also $K \subset \bigcup_{i=1}^N B_{r_i}(y_i)$, we have $B_{r_1} \cap K = \emptyset$, or $B_{r_1} \subset K^c$, so there is an open ball around x that is contained in K, for arbitrary $x \in K$. That completes the proof.

Exercise 2 Let $f : \mathbb{R} \to \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exists $L > 0$ such that

$$|f(s,x) - f(s,y)| \le L|x - y|,$$

for all $x, y \in \mathbb{R}$. Prove that there exists $\epsilon > 0$ such that the initial value problem

$$\begin{cases} y'(t) & = f(t, y(t)) \\ y(0) & = y_0 \end{cases}$$

has a unique solution for $t \in (-\epsilon, \epsilon)$. For, follow the steps:

(i) For each $\epsilon > 0$, consider the map $F : C^0([-\epsilon, \epsilon]) \to C^0([\epsilon, \epsilon])$ defined as

$$(F(y))(t) := y_0 + \int_0^t f(s, y(s))ds, \quad t \in [-\epsilon, \epsilon]$$

Prove that it is possible to choose $\epsilon > 0$ such that the above map satisfies the assumption of the **Banach Fixed Point Theorem**;

(ii) Prove that a fixed point of F is a solution to the initial value problem.

Proof

(i) We have to show that there is an $\alpha \in [0, 1)$ such that we have the approximation:

$$|F(u) - F(v)|_{C^0} \leq |u - v|_{C^0}, \forall u, v \in C^0([-\epsilon, \epsilon])$$

This should hold by picking $\epsilon > 0$ sufficiently small the following approximation:

$$\begin{aligned} |F(u)(t) - F(v)(t)| &= \left| \left(y_0 + \int_0^t f(s, u(s))ds \right) - \left(y_0 + \int_0^t f(s, v(s))ds \right) \right| \\ &= \left| \int_0^t f(s, u(s)) - f(s, v(s))ds \right| \\ &\leq \int_0^t |f(s, u(s))ds - f(s, v(s))|ds \end{aligned}$$

Since $f(s, \cdot)$ is Lipschitz, say with constant L, which by the premise does not depend on $s \in [-1, 1]$, $|f(s, u(s)) - f(s, v(s))| \leq L|u(s) - v(s)| \leq L|u - v|_{C^0}$, Therefore

$$|F(u)(t) - F(v)(t)| \leq \int_0^t L|u - v|_{C^0} ds = Lt|u - v|_{C^0} \leq L\epsilon|u - v|_{C^0}$$

And taking the supremum over $t \in [-\epsilon, \epsilon]$ yields $|F(u) - F(v)|_{C^0} \leq L\epsilon|u - v|_{C^0}$. If we let $\epsilon < \frac{1}{L}$, then we see that $F : C^0([-\epsilon, \epsilon]) \to C^0([-\epsilon, \epsilon])$ is a contraction. Since the target space of $C^0([-\epsilon, \epsilon], \mathbb{R}$ is $\mathbb{R}$, which is complete, it follows that any Cauchy sequence of functions $(f_n)_{n \in \mathbb{N}} \subset C^0([-\epsilon, \epsilon])$ has a pointwise limit f defined as $f(x) := \lim_{n \to \infty} f_n(x)$. By continuity of any norm, $|f - f_n|_{C^0} = \lim_{m \to \infty} |f_m - f_n|_{C^0} \leq |f_m - f_n|\varepsilon$ if we pick $n, m \geq N$ for an N sufficiently large and use that $(f_n)_{n \in \mathbb{N}}$ is Cauchy. Hence, $f_n \to f$ uniformly. But since uniform convergence preserves continuity, this means $f \in C^0([-\epsilon, \epsilon])$, so $C^0([-\epsilon, \epsilon])$ is a Banach space. This means that $F : C^0([-\epsilon, \epsilon]) \to C^0([-\epsilon, \epsilon])$ is a contraction on a Banach space, and therefore has a unique fixed point, say $y \in C^0([-\epsilon, \epsilon])$.

(ii) We have that $F(y) = y$, hence $y(t) = y_0 + \int_0^t f(s, y(s))ds$. Note that $y \in C^0$, so it is continuous. By the fundamental theorem of calculus,

$F(y) : t \mapsto \int_0^t f(s, y(s))ds$ is differentiable and has derivative $F(y)'(t) = f(t, y(t))$. Now, since f is continuous by the premise, and $y \in C^0$ and $t \mapsto t$ are also continuous, and the composition of continuous functions is continuous, it follows that $t \mapsto f(t, y(t))$ is continuous. So this means that $F(y)$ is of class $C^1([-\epsilon, \epsilon])$, and therefore $y = F(y)$ is C^1. Moreover, if we differentiate with respect to t on both sides of the equality $y(t) = F(y)(t)$, it reads by the fundamental theorem of calculus:

$$y'(t) = f(t, y(t))$$

Also,

$$y(0) = y_0 + \int_0^0 f(s, y(s))ds = y_0 + 0 = y_0$$

Therefore, y is a solution of the initial value problem.

Exercise 3 In this exercise we study the space $C_c(\mathbb{R}^N)$ of continuous functions $f : \mathbb{R}^N \to \mathbb{R}$ with compact support. Namely, such that, for each $f \in C_c(\mathbb{R}^N)$, there exists $r > 0$ for which $f = 0$ outside $B(0, r)$. Then:

(i) Prove that the space $C_c(\mathbb{R}^N)$ is not complete with respect to the supremum norm;

(ii) Prove that the closure of the space $C_c(\mathbb{R}^N)$ in the supremum norm, denoted by $C_0(\mathbb{R}^N)$, is the set of functions $f : \mathbb{R}^N \to R$ satisfying the following property: for each $\epsilon > 0$, there exists $r > 0$ such that $|f(x)| < \epsilon$ for all $x \in \mathbb{R}^N$ with $|x| > r$.

Proof

(a) Consider the sequence of functions $(f_n)_{n \geq 5}$ defined as:

$$f_n(x) = \begin{cases} e^{-|x|} & |x| < n \\ e^{-n} - (|x| - n) & n \leq |x| < n + e^{-n} \\ 0 & |x| \geq e^{-n} + n \end{cases}$$

The reason to take this function is that $e^{-|x|} \to 0$ monotonically as $|x| \to \infty$, so that for $f(x) = e^{-|x|}$, the supremum norm $|f_n - f|_{C^0}$ is attained at the boundary of the support $B_n(0)$: it equals $|f_n - f|_{C^0} = e^{-n}$, since we take $n \geq 5$, i.e. n sufficiently large so that the linear segment $e^{-n} - (|x| - n)$ decreases faster than $e^{-|x|}$. We see that therefore $f_n \to f$ uniformly. But f does not have a compact support: it is nonzero on the whole of $\mathbb{R}^N$. On the other hand, because f_n is convergent in C^0, say, it is Cauchy. This implies that $C_c(\mathbb{R}^N)$ is not complete.

(b) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $C_c(\mathbb{R}^N)$, then since it is contained in $C^0(\mathbb{R}^N)$, and $C^0(\mathbb{R}^N)$ is complete, the sequence converges to some function $f \in C^0(\mathbb{R}^N)$. We just need to show that f is contained in $C_0(\mathbb{R}^N)$, and that on the other hand, any function $f \in C_0(\mathbb{R}^N)$ can be approximated uniformly by functions of compact support.

For the first statement, let $\epsilon > 0$. Then there is an $M \in \mathbb{N}$ such that $\forall n \geq M : |f_n - f|_{C^0} < \epsilon$. Take f_M: it has a support contained in a ball with radius $r > 0$. So for $|x| > r$, we have $f_M(x) = 0$, and also $|f(x) - f_M(x)| < \epsilon$, hence $|f(x)| < \epsilon$ for $|x| > r$.

For the second statement, Let $f \in C_0(\mathbb{R}^N)$. We have to show a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c(\mathbb{R}^N)$ that converges to f uniformly. For this, consider the *continuous truncation* f_n:

$$f_n(x) = \begin{cases} f(x) & x \in B_n(0) \\ f\left(\frac{nx}{|x|}\right)\frac{n-|x|}{2n} & x \in B_{2n}(0)\backslash B_n(x) \\ 0 & |x| \geq 2n \end{cases}$$

We indeed see that $|f_n - f|_{C^0} \leq |f(n)|$, which goes to 0 as $n \to \infty$ since $\forall\epsilon > 0$ we can find an $M \in \mathbb{N}$ such that for all $|x| > M$, $|f(x)| < \epsilon$. So pick M sufficiently large, then $|f_n - f|_{C^0}$ is made arbitrarily small. So $C_0(\mathbb{R}^N) = \overline{C_c(\mathbb{R}^N)}^{C^0}$, i.e. every $f \in C_0$ is a limit of a C_c-sequence, and every C_C-sequence converges in C_0.