Analysis 2, Chapter 12

Matthijs Muis

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1 Lebesgue Measure

1.1 Definitions

Definition 1. *An* ***open cube*** *of* ***width*** r ***centered at a point*** $x \in \mathbb{R}^N$*, denoted* $Q_r(x)$ *is the set*

$$Q_r(x) := x + Q_r$$

Where $Q_r = Q_r(0)$ *is defined as:*

$$Q_r := \left(-\frac{r}{2}, \frac{r}{2}\right)^N$$

For a ***closed cube****, the definition is the same, but take closed intervals.*

Any **cube** is a **rectangle**:

Definition 2. *An* ***open rectangle*** *of* ***dimensions*** $r_1, ...r_N$ ***centered at a point*** $x \in \mathbb{R}^N$*, is the set*

$$x + \prod_{i=1}^{N} \left[-\frac{r_i}{2}, \frac{r_i}{2}\right]^N$$

For a ***closed rectangle****, the definition is the same, but take closed intervals.*

Definition 3. *The* ***volume*** $\mathsf{vol}(R)$ *of a* ***rectangle*** $R = x + \prod_{i=1}^{N} \left[-\frac{r_i}{2}, \frac{r_i}{2}\right]^N$ *is defined as:*

$$\mathsf{vol}(R) := \prod_{i=1}^{N} r_i$$

Definition 4. *Let* $E \subset \mathbb{R}^N$*. We define the* ***Lebesgue outer measure*** *of* E *as*

$$\mathcal{L}(E) = \inf \left\{ \sum_{n=1}^{\infty} r_n^N : \{Q_{r_n}(x_n)\}_{i=1}^{\infty} \text{ covers } E \right\}$$

Remark 1. *We can equivalently put "open rectangles" in this definition everywhere where it reads "open cubes". This is because an open cube is also an open rectangle, and the other way around, we can approximate the* ***volume*** *of an open* *rectange R with the sums of **volumes** of a sequence of finite coverings in open cubes of R, therefore we can replace any R_i in an open rectangle-covering of E with a finite number of cubes $Q_1^i, ..., Q_n^i$ such that $\sum_k Q\mathsf{vol}(Q_k^i) = \mathsf{vol}(R) + \epsilon/2^i$ for $\epsilon > 0$ arbitrarily small, thereby we can approximate the sum of the volumes in the rectangle covering up to 2ϵ-precision, which means that the infimum over all cube coverings should equal the infimum over all rectangle coverings.*

Lemma 1. *If $R = \prod_{i=1}^N (a_i, b_i)$ is an open rectangle,*

$$\mathcal{L}(R) = \prod_{i=1}^{N} |b_i - a_i|$$

Lemma 2. *If R, S are rectangles (open, closed, or half-open for some intervals), then*

$$\mathcal{L}(R) = \mathcal{L}(R \cap S) + \mathcal{L}(R \backslash S)$$

Lemma 3. *(i) $\mathcal{L}(\emptyset) = 0$*

*(ii) **monotonicity**: If $E \subset F$ then $\mathcal{L}(E) \leq \mathcal{L}(F)$*

*(iii) **countable subadditivity**: For any countable collection of sets $\{E_n\}_{n\in\mathbb{N}} \subset 2^{\mathbb{R}^N}$, it holds:*

$$\mathcal{L}\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathcal{L}(E_i)$$

*(iv) **translation invariance**: for all $E \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$: $\mathcal{L}(x+E) = \mathcal{L}(E)$*

Lemma 4. *For $\{Q_i\}_{i\in\mathbb{N}}$ a countable collection of cubes with **pairwise disjoint interiors**, it holds*

$$\mathcal{L}\left(\bigcup_{i=1}^{\infty} Q_i\right) = \mathcal{L}\left(\bigcup_{i=1}^{\infty} \overline{Q_i}\right) = \sum_{i=1}^{\infty} \mathcal{L}(Q_i) = \sum_{i=1}^{\infty} \mathsf{vol}(Q_i)$$

Remark 2. *One of the benefits of using the Lebesgue outer measure over the **Peano-Jordan content** to measure sets, is that unbounded sets such as $\mathbb{Q}$, can have finite measure. We cannot cover $\mathbb{Q}$ with a finite collection of cubes, so its content is ∞.*

*But we can enumerate $\mathbb{Q}$ as $\{q_i\}_{i\in\mathbb{N}}$, then give every rational q_i its own $\frac{\epsilon}{2^i}$-cube and conclude that for any $\epsilon > 0$, we have a cover of $\mathbb{Q}$ of total volume $\epsilon = \sum_{i\in\mathbb{N}} \frac{\epsilon}{2^i}$. Therefore, $\mathbb{Q}$ is even **$\mathcal{L}$-negligible**.*

Lemma 5. ***Open** sets $E \subset \mathbb{R}^N$ can be written as the countable union of **pairwise disjoint closed cubes**.*

Remark 3. *For the open cube $(0,1)^N$, such a union would be countable, namely $\bigcup_{n=1}^{\infty} [\frac{1}{n}, \frac{n-1}{n}]^N$*

1.2 Measurable sets

Definition 5. *A set $\subset \mathbb{R}^N$ is called* ***Lebesgue measurable****, or* ***$\mathcal{L}$-measurable*** *if for any $F \subset \mathbb{R}^N$, we have:*

$$\mathcal{L}(F) = \mathcal{L}(F \cap E) + \mathcal{L}(F \backslash E)$$

Lemma 6. *If F, D are sets such that*

$$d(D, F) := \inf \{|x - y| : x \in D, y \in F\} > 0$$

Then $\mathcal{L}(F \cup D) = \mathcal{L}(F) + \mathcal{L}(D)$

Lemma 7. *Let $E \subset \mathbb{R}^N$. Then, E is Lebesgue measurable if and only if*

$$\mathcal{L}(G \cup F) = \mathcal{L}(G) + \mathcal{L}(F)$$

, for all $G \subset E$, and all $F \subset \mathbb{R}^N$ with $E \cap F = \emptyset$

Proposition 1. *The following hold:*

(i) $\emptyset$ and $\mathbb{R}^N$ are Lebesgue measurable;

(ii) If $E, F \subset \mathbb{R}^N$ are Lebesgue measurable, then also $E \backslash F$ is;

(iii) If $(E_i)i \in \mathbb{N} \subset \mathbb{R}^N$ is a sequence of Lebesgue measurable sets, then

$$\bigcup_{i \in \mathbb{N}} E_i, \quad \bigcap_{i \in \mathbb{N}} E_i$$

are also Lebesgue measurable;

(iv) If $E \subset \mathbb{R}^N$ is Lebesgue measurable, then $x + E$ is Lebesgue measurable, for all $x \in \mathbb{R}^N$

Families of sets that are closed under complement, countable union and countable intersection are called σ-algebras

Definition 6. *Let Ω be a set. A family of sets $\mathcal{A} \subset 2^\Omega$ is called a σ-algebra over Ω if*

(i) $\emptyset \in \mathcal{A}$;

(ii) If $E \in A$, then $\Omega \backslash E \in A$;

(iii) If $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$, then

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$$

Remark 4. *Therefore, by* ***de Morgan's laws****, if $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$, then*

$$\bigcap_{i=1}^{\infty} E_i = \Omega \backslash \bigcup_{i=1}^{\infty} (\Omega \backslash E_i) \in \mathcal{A}$$

Proposition 2. *Open rectangles $R \subset \mathbb{R}^N$ are Lebesgue measurable.*

Proof. We will do the proof because it is quite essential: for any two sets $R, F \subset \mathbb{R}^N$, we can use subadditivity:

$$\mathcal{L}(A) + \mathcal{L}(B) \geq \mathcal{L}(A \cup B)$$

Which follows since any countable cover $\mathcal{C}_A$ of A can together with any countable cover $\mathcal{C}_B$ of B be united to a countable cover $\mathcal{C}_A \cup \mathcal{C}_B$ for $A \cup B$, and for the total volume, which we define over countable collections of cubes as:

$$\mathsf{vol}(\mathcal{C}) = \sum_{Q \in \mathcal{C}} \mathsf{vol}(Q)$$

It holds

$$\mathsf{vol}(\mathcal{C}_A) + \mathsf{vol}(\mathcal{C}_B) \geq \mathsf{vol}(\mathcal{C}_A \cup \mathcal{C}_B) \geq \inf_{\mathcal{C}: \cup \mathcal{C} \supset A \cup B} \mathsf{vol}(\mathcal{C}) = \mathcal{L}(A \cup B)$$

So when we take the **infinimum** on the right, we get $\mathcal{L}(A) + \mathcal{L}(B) \geq \mathcal{L}(A \cup B)$.

From this, we get one inequality:

$$\mathcal{L}(R \cap F) + \mathcal{L}(F \backslash R) \geq \mathcal{L}(F), \text{ where we use } A = R \cap F,\ B = F \backslash R$$

The other inequality is not for every R: in the case of open rectangles, we can derive it because the **boundary of** R is **sufficiently regular**.

Let $\{Q_i\}_{i \in \mathbb{N}}$ be a covering of open cubes for F such that $\sum_{i=1}^{\infty} \mathsf{vol}(Q_i) < \mathcal{L}(F) + \epsilon$. Define

$$B_i = Q_i \cap R \quad A_i = Q_i \backslash R, \text{ then } R \cap F \subset \bigcup_{i=1}^{\infty} B_i, \quad R \backslash F \subset \bigcup_{i=1}^{\infty} A_i$$

These are rectangles, by being the intersection of two rectangles. We already saw for rectangles D and P that

$$\mathcal{L}(D \cap P) + \mathcal{L}(D \backslash D) = \mathcal{L}(D), \text{ therefore } \mathcal{L}(A_i) + \mathcal{L}(B_i) = \mathcal{L}(Q)$$

Since the covers $\{A_i\}_{i=1}^{\infty}$, $\{B_i\}_{i=1}^{\infty}$ do not have any equal cubes,

$$\mathsf{vol}(\{A_i\}_{i=1}^{\infty}) + \mathsf{vol}(\{B_i\}_{i=1}^{\infty}) = \mathsf{vol}(\{A_i, B_i\}_{i=1}^{\infty})$$

Now, A_i is not necessarily a collection of open cubes, which is a bureaucratic detail that requires a workaround. We can consider, for each A_i, an open rectangle $O_i \supset A_i$ with $\mathsf{vol}(O_i) < \mathsf{vol}(A_i) + \frac{1}{2^i}$, which gives a cover in open cubes with a total volume of $\mathsf{vol}(\{O_i\}_{i=1}^{\infty}) < \mathsf{vol}(\{A_i\}_{i=1}^{\infty}) + \epsilon$, and this will still not contain any duplicates with $\{B_i\}_{i=1}^{\infty}$, therefore:

$$\begin{aligned}
\mathcal{L}(F \backslash R) + \mathcal{L}(F \cap R) &\leq \mathsf{vol}(\{O_i\}_{i=1}^{\infty}) + \mathsf{vol}(\{B_i\}_{i=1}^{\infty}) \\
&< \mathsf{vol}(\{O_i\}_{i=1}^{\infty}) + \epsilon + \mathsf{vol}(\{B_i\}_{i=1}^{\infty}) \\
&= \mathsf{vol}(\{A_i, B_i\}_{i=1}^{\infty}) + \epsilon \\
&\leq \mathcal{L}(F) + \epsilon + \epsilon
\end{aligned}$$

So for any $\epsilon > 0$, it is possible to conclude

$$\mathcal{L}(F \backslash R) + \mathcal{L}(F \cap R) \leq \mathcal{L}(F) + 2 \cdot \epsilon$$

Which gives the required opposite inequality.

□

Definition 7. *A set $E \subset \mathbb{R}^N$ is said to be a* ***Borel set*** *if it can be written as countable union and countable intersection of open sets.*

Lemma 8. *Borel sets are Lebesgue measurable.*

Proof. We already have Lebesgue measurability of (closed) rectangles. We also know that every open set is a countable union of closed cubes, and since the union of countable Lebesgue measurable sets is Lebesgue measurable, we get that every open set is open. Using that measurability is preserved under countable union and intersection and taking complements, this means every Borel set is measurable. □

Finally, some equivalent characterizations of measurability,

Proposition 3. *The following are equivalent for $E \subset \mathbb{R}^N$:*

(i) E is Lebesgue measurable;

(ii) For every $\epsilon > 0$ there exists an open set $U \supset E$ such that $\mathcal{L}(U \backslash E) < \epsilon$;

(iii) For every $\epsilon > 0$ there exists a closed set $C \subset E$ such that $\mathcal{L}(E \backslash C) < \epsilon$.

1.3 Negligible Sets

Definition 8. *We say that a set $E \subset \mathbb{R}^N$ is $\mathcal{L}$* ***-negligible*** *(or* ***negligible****) with respect to the Lebesgue measure) if $\mathcal{L}(E) = 0$.*

Lemma 9. *Let $E \subset \mathbb{R}^N$. Then, following are equivalent:*

(i) For every $\epsilon > 0$, there exists $\{E_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^N$ such that

$$E \subset \bigcup_{i=1}^{\infty} E_i \text{ and } \mathcal{L}(\bigcup_{i=1}^{\infty} E_i) < \epsilon$$

(ii) E is $\mathcal{L}$-negligible.

Lemma 10. *Let $E \subset \mathbb{R}^N$ be $\mathcal{L}$ -negligible. Then, E is $\mathcal{L}$ -measurable.*

Definition 9. *We say that a property holds for* ***$\mathcal{L}$-almost every*** *$x \in \mathbb{R}^N$ (****$\mathcal{L}$ -a.e. in*** *$\mathbb{R}^N$) if it holds for all $x \in \mathbb{R}^N \backslash N$, where $N \subset \mathbb{R}^N$ is $\mathcal{L}$* ***-negligible***

1.4 Operations on Measurable sets

Definition 10. *We call a sequence of sets* $(E_i)_{i=1}^{\infty}$*:*

(i) ***increasing*** *if* $\forall i \in \mathbb{N} : E_i \subset E_{i+1}$

(ii) ***decreasing*** *if* $\forall i \in \mathbb{N} : E_i \supset E_{i+1}$

(iii) ***monotone*** *if it is either increasing or decreasing.*

Definition 11. *For an increasing sequence of sets* $(E_i)_{i=1}^{\infty}$*, we define a limiting set:*

$$\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$$

We also denote $E_i \uparrow E$ *for* $E = \bigcup_{i\in\mathbb{N}} E_i$*. For a decreasing sequence of sets* $(E_i)_{i=1}^{\infty}$*, we define a limiting set:*

$$\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$$

We also denote $E_i \downarrow E$ *for* $E = \bigcap_{i\in\mathbb{N}} E_i$*. In general, we can define limes infinimum and limes supremum:*

$$\liminf_{n\to\infty} E_i = \bigcap_{n\geq 0} \bigcup_{k\geq n} E_k \quad \limsup_{n\to\infty} E_i = \bigcap_{n\geq 0} \bigcup_{k\geq n} E_k$$

The lim inf and lim sup are quite natural definitions, since:

1. $\bigcap \mathcal{C}$ is the largest set (maxim**um** with respect to inclusion $\subset$) that is contained in all $S \in \mathcal{C}$. Therefore, it is a $\inf \mathcal{C}$ with respect to inclusion.

2. $\left(\bigcap_{k\geq n} E_k\right)_{n\in\mathbb{N}}$ is a **decreasing sequence of sets**, therefore we can naturally define its limit. So, in this way:

$$\liminf_{n\to\infty} E_n = \lim_{n\to\infty} \inf_{k\geq n} E_k$$

3. $\bigcup \mathcal{C}$ is the smallest set (minim**um** with respect to inclusion $\subset$) that contains all $S \in \mathcal{C}$. Therefore, it is a $\sup \mathcal{C}$ with respect to inclusion.

4. $\left(\bigcap_{k\geq n} E_k\right)_{n\in\mathbb{N}}$ is a **decreasing sequence of sets**, therefore we can naturally define its limit. So, in this way:

$$\liminf_{n\to\infty} E_n = \lim_{n\to\infty} \sup_{k\geq n} E_k$$

Proposition 4. *The following hold for* $\mathcal{L}$*-measurable sets* E, F *and sequences of* $\mathcal{L}$*-measurable sets* $(E_i)_{i\in\mathbb{N}}$*:*

(i) σ-***additivity***: *if* $E_j \cap E_j = \emptyset$ *for all* $i \neq j \in \mathbb{N}$*, then*

$$\mathcal{L}(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathcal{L}(E_i)$$

(ii) ***upward continuity***: *If* $(E_i)_{i\in\mathbb{N}}$ *is* ***ascending****, and* $E_i \uparrow E$ *then*

$$\mathcal{L}(E) = \lim_{i\to\infty} \mathcal{L}(E_i)$$

(iii) ***downward continuity***: *If* $(E_i)_{i\in\mathbb{N}}$ *is* ***descending****, and* $E_i \downarrow E$*, and* $\mathcal{L}(E_1) < \infty$ *then*

$$\mathcal{L}(E) = \lim_{i\to\infty} \mathcal{L}(E_i)$$

(iv) *If* $E \subset F$*, then* $\mathcal{L}(F) = \mathcal{L}(E) + \mathcal{L}(F \backslash E)$

Be mindful of the extra condition $\mathcal{L}(E_1) < \infty$ in downward continuity: for a counterexample, see

$$E_i = (i, \infty)$$

Then $E_i \downarrow \emptyset$, yet $\lim_{i\to\infty} \mathcal{L}(E_i) = \lim_{i\to\infty} \infty$ because each E_i has measure ∞ (note that $\mathcal{L} : 2^{\mathbb{R}^N} \to [0, +\infty]$ is defined to take infinite values if necessary).

2 Homework

Exercise 1 Let $R \subset \mathbb{R}^N$ be a rectangle:

$$R = [a_1, b_1] \times ... \times [a_N, b_N]$$

Prove that $\mathcal{L}(R) = (b_1 - a_1)...(b_N - a_N)$

Proof We start that by noting that $R \subset R$ and $\mathsf{vol}(R) = (b_1 - a_1)...(b_N - a_N)$, thereby

$$\mathcal{L}(R) = \inf\{\sum_{i\in\mathbb{N}} \mathsf{vol}(R_i) : \{R_i\}_{i\in\mathbb{N}} \text{ covers } R\} \leq \mathsf{vol}(R)$$

Since $\{R\}$ is a **cover** of R. For the other inequality, suppose that we have a countable *cube*-cover $\{Q_{r_i}(x_i)\}_{i\in\mathbb{N}}$ for R. By compactness of R, which is a closed and bounded subset of $\mathbb{R}^N$ (so use Bolzano-Weierstrass to conclude compactness), we can extract a finite subcover $\{Q_{r_i}(x_i)\}_{i=1,...,k}$, which we immediately relabel to have indices $1, ...k$. By the usual properties of volume, we have

$$\text{if } R \subset \bigcup_{i=1}^{k} Q_{r_i}(x_i) \text{ then } \mathsf{vol}(R) \leq \sum_{i=1}^{k} \mathsf{vol}(Q_{r_i}(x_i))$$

This implies that for any countable cube-cover $\{Q_{r_i}(x_i)\}_{i\in\mathbb{N}}$ for R, we have

$$\sum_{i=1}^{\infty} \mathsf{vol}(Q_{r_i}(x_i)) \geq \sum_{i=1}^{k} \mathsf{vol}(Q_{r_i}(x_i)) \geq \mathsf{vol}(R)$$

Therefore, if we take inf on both sides, we read $\mathcal{L}(R) \geq \mathsf{vol}(R)$.

The other homework exercises remain handwritten.