Analysis 2, Chapter 11

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1 Darboux & Riemann Integration

1.1 Peano-Jordan Content

Definition 1. *A set $R \subset \mathbb{R}^N$ is called a **pluri-rectangle** if it is possible to write it as a finite union of **closed rectangles**:*

$$R = \bigcup_{i=1}^{k} R_i$$

*where each $R_i \subset \mathbb{R}^N$ is a **closed rectangle**.*

Lemma 1. *We can always assume the R_i to have **pairwise disjoint interiors**. That is, we can rewrite any pluri-rectangle $R = \bigcup_{i=1}^{k} R_i$ as*

$$R = \bigcup_{i=1}^{m} P_i$$

Where $P_i^o \cap P_j^o = \emptyset$ for all $1 \leq i \neq j \leq m$

Definition 2. ***The Peano-Jordan Content of a pluri-rectangle***

*If R is a **rectangle**, we define its **Peano-Jordan content** as:*

$$\mathcal{PJ}(R) = \mathsf{vol}(R)$$

*If R is a **pluri-rectangle** $R = \bigcup_{i=1}^{k} R_i$, where we assume the $\mathbb{R}_i^o$ to be pairwise disjoint as in the lemma, we define its Peano-Jordan as:*

$$\mathcal{PJ}(R) = \sum_{i=1}^{k} \mathsf{vol}(R_i)$$

Definition 3. *For $E \subset \mathbb{R}^N$, we define the **inner Peano-Jordan content** as:*

$$\mathcal{PJ}^-(E) = \sup \{\mathcal{PJ}(R) \mid R \subset E,\ R \text{ a pluri-rectangle}\}$$

*and the **outer Peano-Jordan content** as*

$$\mathcal{PJ}^+(E) = \inf\{\mathcal{PJ}(R) \mid E \subset R,\ R \text{ a pluri-rectangle}\}$$

We say that a set $E \subset \mathbb{R}^N$ is ***Peano-Jordan measurable*** *if*

$$\mathcal{PJ}^+(E) = \mathcal{PJ}^-(E)$$

And we define its ***Peano-Jordan measure*** *as $\mathcal{PJ}(E) := \mathcal{PJ}^+(E) = \mathcal{PJ}^-(E)$. Note that if $\mathcal{PJ}^+(E) = \infty$ and $\mathcal{PJ}^-(E) = \infty$, we consider this as $\mathcal{PJ}^+(E) = \mathcal{PJ}^-(E)$ and we define its Peano-Jordan measure as $\mathcal{PJ}(E) = \infty$.*

Lemma 2. *Let$E \subset \mathbb{R}^N$. Then,*

(i) $\mathcal{PJ}^-(E) = 0$ if and only if E has ***empty interior****;*

(ii) Assume that $\mathcal{PJ}+(E) < \infty$. Then, E is ***bounded****;*

(iii) $\mathcal{PJ}+(E) = \mathcal{PJ}+(E)$

So, the outer Peano-Jordan content is *almost* the same as the Lebesgue outer-measure, with the difference that it is only allowed to look at finite covers of E in rectangles. A problem is that unbounded sets will always have **infinite outer Peano-Jordan content**, since we cannot use finitely many bounded rectangles to cover such sets. We will see that this also limits what we can integrate.

As a very simple example, $\mathbb{Q} \subset \mathbb{R}^N$ has empty interior, hence $\mathcal{PJ}(\mathbb{Q}) = 0$, but by unboundedness, $\mathcal{PJ}(\mathbb{Q}) = \infty$. $\mathbb{Q}$, however, is Lebesgue measurable: $\{(q_i - \epsilon/2^i, q + \epsilon/2^i)\}_{i\in\mathbb{N}}$, where we enumerate all rationals q_i with a function $i \mapsto q_i$, $q : \mathbb{N} \to \mathbb{Q}$

Lemma 3. *Let $E_1, ..., E_m \subset \mathbb{R}^N$ be Peano-Jordan measurable sets. Then,*

$$\bigcup_{i=1}^{m} E_i \quad \bigcap_{i=1}^{m} E_i$$

are also Peano-Jordan measurable. Moreover, if $E_1, ..., E_m \subset \mathbb{R}^N$ are pairwise disjoint, then

$$\mathcal{PJ}\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} \mathcal{PJ}(Ei)$$

Proposition 1. *Let $f : \mathbb{R}^N \to R$ be a continuous function and $K \subset \mathbb{R}^N$ compact. Then,*

$$\mathcal{PJ}(\{(x, f(x)) : x \in K\}) = 0$$

Theorem 1. *Let $E \subset \mathbb{R}^N$ be a bounded set. Then,*

$$\mathcal{PJ}^+(E) - \mathcal{PJ}^-(E) = \mathcal{PJ}+(\partial E)$$

In particular, E is Peano-Jordan measurable if and only if $\mathcal{PJ}+(\partial E)$

1.2 Darboux-Riemann Integral

Definition 4. *For $A \subset \Omega$, we define the* ***indicator function*** $\delta_A : 2^A$ *as:*

$$\delta_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Definition 5. *Let $f : D \to R$. We say that f is a* ***piecewise-constant function*** *if*

$$f(x) = \sum_{i=1}^{k} y_i \delta_{E_i}(x)$$

for some $y_i \in \mathbb{R}$, and some rectangles $E_1, ..., E_k \subset \mathbb{R}^N$ which have ***pairwise disjoint interiors*** *(in particular, they can overlap on boundaries, which gives two nonzero terms in the sum for such $x \in E_i \cap E_j$, $j \neq i$), such that*

$$\bigcup_{i=1}^{k} E_i = D$$

. This implies that D is a pluri-rectangle.

Definition 6. *Let $f : D \to R$ be a piecewise-constant function. We define the* ***Darboux integral of*** f ***on*** D *as*

$$\int_D f(x)dx := \sum_{i=1}^{k} y_i \mathcal{PJ}(E_i)$$

Definition 7. *The* ***upper*** *and* ***lower Darboux integrals*** *of any function $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$ is a (pluri)-rectangle, are defined as:*

$$\underline{\int_D} f(x)dx := \sup \left\{ \int_D g(x)dx \mid f \geq g \text{ and } g \text{ piecewise constant} \right\}$$
$$\overline{\int_D} f(x)dx := \inf \left\{ \int_D h(x)dx \mid f \leq h \text{ and } h \text{ piecewise constant} \right\}$$

We say $f : D \to \mathbb{R}$ is ***Darboux integrable*** *if*

$$\underline{\int_D} f(x)dx = \overline{\int_D} f(x)dx$$

and then define its Darboux integral as this value:

$$\int_D f(x)dx = \underline{\int_D} f(x)dx = \overline{\int_D} f(x)dx$$

Theorem 2. *If $f : D \to \mathbb{R}$ is a function on a (pluri)-rectangle D, and we let $R(D) \subset 2^{2^D}$ denote the family of all finite collections of rectangles*

$$\mathcal{C} = \{E_1, ..., E_k\} \subset 2^D$$

that have pairwise disjoint interiors such that $\bigcup_{i=1}^k E_i = D$ Then,

$$\underline{\int_D} f(x)dx := \sup\left\{\sum_{i=1}^k M_i \mathcal{PJ}(E_i) \mid \{E_i\}_{i=1}^k \in R(D),\ M_i = \sup f(E_i)\right\}$$
$$\overline{\int_D} f(x)dx := \inf\left\{\sum_{i=1}^k m_i \mathcal{PJ}(E_i) \mid \{E_i\}_{i=1}^k \in R(D),\ m_i = \inf f(E_i)\right\}$$

And, f is Darboux integrable if and only if for all $\epsilon > 0$, we can find a partition $\{E_i\}_{i=1}^k$ such that

$$\sum_{i=1}^k |M_i - m_i| \mathcal{PJ}(E_i) < \epsilon$$

But we have an alternative characterization that sees the Darboux-Riemann integral as the signed "volume under the graph":

Theorem 3. *Let $f : D \to \mathbb{R}$ where D is a (pluri)-rectangle. Then f is Darboux-integrable if and only if the following sets:*

$$E^+ := \{(x, y) \in D \times [0, \infty) \mid 0 \leq y \leq f(x)\}$$
$$E^- := \{(x, y) \in D \times (-\infty, 0] \mid f(x) \leq y \leq 0\}$$

*Are both **Peano-Jordan measurable**. Moreover, then we have the following way to compute the Darboux integral:*

$$\int_D f(x)dx = \mathcal{PJ}(E^+) - \mathcal{PJ}(E^-)$$