Analysis 2, Chapter 1

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1 Structures on Spaces

Discusses *topological spaces*, *metric spaces*, *normed spaces*, *inner product spaces.* Spaces with a norm or inner product are always vector spaces, since the norm/inner product interacts with addition and scalar multiplication. We will next define convergence in metric spaces and see what this definition translates to in the other spaces.

1.1 Inner Product Spaces

Definition 1. *An* ***inner product space*** *is a tuple* $(X, \langle\cdot,\cdot\rangle)$ *where* X *is an* K*-vector space over* $\mathbb{C}$ *or* $\mathbb{R}$*, and* $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}_{\geq 0}$ *is a* ***scalar product****, a bifunction that satisfies:*

1. ***sequilinearity****:* $\forall \lambda, x, y : \langle \lambda x + y, z\rangle = \lambda\langle x, z\rangle + \langle y, z\rangle$ *and* $\forall \lambda, x, y, z :$ $\langle z, \lambda x + y\rangle = \overline{\lambda}\langle z, x\rangle + \langle z, y\rangle$
2. ***sequisymmetry****:* $\forall x, y, z : \langle x, y\rangle = \overline{\langle y, z\rangle}$
3. ***definiteness****:* $\forall x : \langle x, x\rangle \geq 0$ *and* $\forall x : \langle x, x\rangle = 0 \iff x = 0$*. Note that this is well-defined since* $\langle x, x\rangle \in \mathbb{R}$ *is by sequisymmetry.*

Proposition 1. *In scalar product spaces, the following identities hold:*

1. ***the Pythagorean theorem:***

$$\langle v, w\rangle = 0 \implies \langle v - w, v - w\rangle = \langle v, v\rangle + \langle w, w\rangle$$

2. ***the parallelogram law:***

$$2\langle v, v\rangle + 2\langle w, w\rangle = \langle v - w, v - w\rangle + \langle v + w, v + w\rangle$$

3. ***the Cauchy-Schwarz inequality:***

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

1.2 Normed spaces

Definition 2. *A **normed space** is a tuple $(X, |\cdot|)$, where X is an K-vector space and $|\cdot| : X \to \mathbb{R}_{\geq 0}$ is a **norm**, a function that satisfies:*

1. ***definiteness*** $\forall x : |x| = 0 \iff x = 0$
2. ***triangle inequality*** $\forall xy : |x + y| \leq |x| + |y|$
3. ***homogeneity*** $\forall \lambda \in K : \forall x : |\lambda x| = |\lambda||x|$

Proposition 2. *If $(X, \langle \cdot, \cdot \rangle)$ is a scalar product, then $|\cdot| : X \to \mathbb{R}$ defined through*

$$|x| := \langle x, x \rangle^{\frac{1}{2}}$$

is a norm.

Remark 1. *A norm gives a notion of length, while an inner product gives a notion of angle, or similarity.*

Proposition 3. *Given a normed space $(X, |\cdot|)$, if the parallelogram law*

$$\forall vw : 2|v|^2 + 2|w|^2 = |v + w|^2 + |v - w|^2$$

holds, then there is a scalar product that induces that norm. In particular, it is given by

$$\langle v, w \rangle = \frac{1}{4}(|v + w|^2 - |v - w|^2)$$

And the parallellogram law for the inner product reads:

$$\forall vw : 2\langle v, v \rangle + 2\langle w, w \rangle = \langle v + w, v + w \rangle + \langle v - w, v - w \rangle$$

Definition 3. *The **unit ball of a norm** $|\cdot|$, $B_{|\cdot|}$, is defined as the set*

$$B_{|\cdot|} = \{x \in X : |x| < 1\}$$

Lemma 1. *For any two norms $|\cdot|_1, |\cdot|_2$, we have:*

$$|\cdot|_1 = |\cdot|_2 \iff B_{|\cdot|_1} = B_{|\cdot|_2}$$

Lemma 2. *In any normed space $(X, |\cdot|)$, $B_{|\cdot|}$ is convex and symmetric with respect to the origin:*

$$\forall 0 \leq \lambda \leq 1, x, y \in B_{|\cdot|} : \lambda x + (1 - \lambda) y \in B_{|\cdot|}$$
$$\forall x : x \in B_{|\cdot|} \iff -x \in B_{|\cdot|}$$

Proposition 4. *In $\mathbb{R}^N$, given $B \subset \mathbb{R}^N$, a **bounded** set that is*

1. ***convex:*** $\forall x, y \in B, \lambda \in [0, 1] : \lambda x + (1 - \lambda) y \in B$

2. ***symmetric with respect to*** 0*, i.e.* $\forall x : x \in B \iff -x \in B$

3. ***not contained*** *in any* k*-dimensional linear space with* $k < N$

There exists a unique norm $|\cdot|$ *so that* B *is the closure of its unit ball* $B_{|\cdot|}$*. In particular, it can be constructed as:*

$$|x| = \min\{\lambda > 0 : \frac{1}{\lambda}x \in B\}$$

We can define the **crystalline norms** this way, for example. These are norms defined as the unique norm arising from a polytope B that is convex and symmetric with respect to 0.

1.3 Metric spaces

Definition 4. *A* ***metric space*** *is a pair* (X, d) *where* $d : X \times X \to \mathbb{R}_{\geq 0}$ *is called a metric and satisfies:*

1. ***positivity:*** $\forall x, y \in X : d(x, y) = 0 \iff x = y$

2. ***symmetry:*** $\forall x, y \in X : d(x, y) = d(y, x)$

3. ***triangle inequality:*** $\forall x, y, z \in X : d(x, y) \leq d(x, z) + d(y, z)$

Lemma 3. *If* $(X, |\cdot|)$ *is a normed space, then* $d : X \times X \to \mathbb{R}$ *through* $d(x, y) = |x - y|$ *is a metric.*

We can also say something about the converse for metrics:

Proposition 5. *For* (X, d) *a metric space and* X *a vector space,* d *comes from a norm if and only if it is:*

1. ***homogenous:*** $\forall \lambda, x, y : d(\lambda x, \lambda y) = |\lambda| d(x, y)$

2. ***translation invariant:*** $\forall x, y, z : d(x + z, y + z) = d(x, y)$

Lemma 4. *In an inner product space, this reads as* $d(x, y) = \langle x - y, x - y\rangle^{\frac{1}{2}}$

Lemma 5. *If* (X, d) *is a metric space, and* $A \subset X$*, then* $(A, d|_A)$ *is a metric space.*

Proposition 6. *for* X, Y *metric spaces with* d_X, d_Y*, we can make* $X \times Y$ *into a metric space with metric* d_{XY} *by setting*

$$d_{XY}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

The so-called product metric.

Remark 2. *Given the Euclidean metric* $x, y \mapsto |x - y|$ *on* $\mathbb{R}$*, where* $|\cdot|$ *is the absolute value, repeating product metric as defined above, will give the* ***Manhattan distance*** *on* $\mathbb{R}^n$*.*

Proposition 7. *For countable products of metric spaces (X_i, d_i), $i \in \mathbb{N}$, we can also construct a metric on $\prod_{i=1}^{\infty} X_i$ as follows:*

$$d(x,y) = \sum_{i\in\mathbb{N}} \frac{d_i(x_i,y_i)}{2^i[1+d_i(x_i,y_i)]}$$

Definition 5. *For (X,d) a metric space, $x \in X$ and $r > 0$, we define the **open ball** of center x and radius r as:*

$$B_r(x) = \{y \in X : d(x,y) < r\}$$

Definition 6. *A set $A \subset X$ is said to be **open** when*

$$\forall x \in A : \exists r > 0 : B_r(x) \subset A$$

Equivalently, if there is a collection $\mathcal{C}$ of open balls such that $\bigcup \mathcal{C} = A$

Definition 7. *A set $C \subset X$ is called **closed** if its complement in X, $C^c = XC$, is open.*

Lemma 6. *Defining open and closed sets in this way gives open and closed sets the following properties: if we let $\mathcal{T}$ denote the collection $\mathcal{T} \subset 2^X$ of all open sets, we have*

1. *$\emptyset$ and X are open.*
2. *$\mathcal{T}$ is closed under unions : $\mathcal{C} \subset \mathcal{T} \implies \bigcup \mathcal{C} \in \mathcal{T}$.*
3. *$\mathcal{T}$ is closed under finite intersection: $O_1, ..., O_k \in \mathcal{T} \implies \bigcap_{i=1}^{k} O_i \in \mathcal{C}$*

We will now proceed with the concept of metric convergence before revisiting the above lemma with the definition of a *topological space*.

1.4 Convergence in metric spaces

Definition 8. *A sequence $(x_n)_{n\in\mathbb{N}}$, (or $(x_n)_{n=0}^{\infty}$, $(x_n)_{n=1}^{\infty}$ is a function $\mathbb{N} \to X$. We also denote $(x_n)_{n\in\mathbb{N}} \subset A$ if $\forall n \in \mathbb{N} : x_n \in A$. This is slight abuse of the subset notation, although one might also call it "overloading $\subset$ for functions".*

Definition 9. *We say that a **real sequence** $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ **converges to** $L \in \mathbb{R}$ if:*

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : |x_n - L| < \epsilon$$

Definition 10. *We say that a sequence $(x_n)_{n=1}^{\infty} \subset X$, metrically converges to an $\overline{x} \in X$ if*

$$\lim_{n\to\infty} d(\overline{x}, x_n) = 0$$

We also use the notation $x_n \to \overline{x}$.

Remark 3. *In a normed space, this reads as $\lim_{n\to\infty} |x_n - \overline{x}| = 0$. And in an inner product space, it reads as $\lim_{n\to\infty}\langle x - x_n, x - x_n\rangle = 0$.*

Lemma 7. *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X,d) that converges to $a \in X$, and to $b \in X$, then $a = b$.*

Definition 11. *A* ***subsequence*** *$u = (x_{n_k})_{k\in\mathbb{N}}$ of a sequence $v = (x_n)_{n\in\mathbb{N}}$ is a sequence $\mathbb{N} \to X$ that can be obtained by the composition $u = v \circ n$ where $n : \mathbb{N} \to \mathbb{N}$ is a function $k \mapsto n_k$ that is* ***strictly increasing****.*

Remark 4. *All sequences over a set, together with the relation $\subset$ meaning "is subsequence of", form a partially ordered set.*

Proposition 8. *If a sequence converges to a limit $l \in X$, then all its subsequences converge, and they converge also to l*

Proposition 9. *The following is known as the* ***Urysohn property****: Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X,d), such that*

$$\exists a \in X : \forall \textit{subsequences } (x_{n_k})_{k\in\mathbb{N}} : \exists \textit{further subsequence } (x_{n_{k_i}})_{i\in\mathbb{N}} : a_{n_{k_i}} \to a$$

Then $x_n \to a$.

Definition 12. *A sequence is called a* ***Cauchy sequence*** *if*

$$\forall \epsilon > 0 \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) < \epsilon$$

Proposition 10. *A convergent sequence is Cauchy.*

Proposition 11. *A Cauchy sequence with a convergent subsequence that converges to $l \in X$ is convergent, namely to l*

Definition 13. *The* ***sequential closure*** *of a set $Y \subset X$ in a metric space (X,d) is defined as:*

$$\overline{Y}^d := \{a \in X : \exists (a_n)_n \subset Y : a_n \to^d a\}$$

1.5 Topological spaces

Definition 14. *A* ***topological space*** *is a pair $(X, \mathcal{T})$, where we call $\mathcal{T} \subset P(X)$ the* ***open*** *sets of X, and $\mathcal{T}$ satisfies:*

1. *$\emptyset$ and X are open.*
2. *$\mathcal{T}$ is closed under unions : $\mathcal{C} \subset \mathcal{T} \implies \bigcup \mathcal{C} \in \mathcal{T}$.*
3. *$\mathcal{T}$ is closed under finite intersection: $O_1, ..., O_k \in \mathcal{T} \implies \bigcap_{i=1}^k O_i \in \mathcal{C}$*

Remark 5. *The mentioned unions can be uncountable.*

Remark 6. *As seen in the lemma, the metric notion of open and closed sets defines a topology on X:*

$$\mathcal{T}_d = \{U \subset X \mid \exists I : \forall i \in I : \exists x_i \in X, r_i > 0 : U = \cup_{i\in I} B_r(x),\}$$

Or, equivalently:

$$\mathcal{T}_d = \{U \subset X \mid \forall x \in U : \exists r > 0 : B_r(x) \subset U\}$$

Definition 15. *We say that a sequence* $(x_n)_{n=0}^{\infty}$ *where* $\forall n \in \mathbb{N} : x_n \in X$, ***topologically converges to*** $\overline{x} \in X$ *if:*

$$\forall U \in \mathcal{T} : \overline{x} \in U : \exists N \in \mathbb{N} : \forall n \geq N : x_n \in U$$

Definition 16. *Let* $Y \subset X$ *where* $(X, \mathcal{T})$ *is a topological space. The* ***topological closure*** *of* Y, $\overline{Y}^{\mathcal{T}}$ *is the smallest closed set (i.e. a complement of an open set) that contains* Y*. This notion coincides with the sequential closure* $\overline{Y}^d$ *in a metric space, if we leet* $\mathcal{T}$ *be induced by the metric* d

In a topological space, the limit of a sequence is not unique. Consider the trivial topology $(X, \mathcal{T} = \{\emptyset, X\})$ on any set X with $|X| \geq 2$, and a periodic sequence in X with two or more values. It is convergent to any of its values.
The reason why the notion of sequential and topological convergence *coincide* in metric spaces, is because for $\overline{x}$, we can find an open set $B_\epsilon(\overline{x})$ around it for any arbitrarily small $\epsilon > 0$. This makes it possible to "distinguish" between any $a, b \in X$ that are not the same: $\epsilon = d(a, b) > 0$, and $B_{\frac{\epsilon}{2}}(a) \cap B_{\frac{\epsilon}{2}}(b) = \emptyset$, so no sequence $(x_n)_{n \in \mathbb{N}}$ can converge to both a and b.
We will treat topological spaces, and what it takes for a topology to be **metrizable**, in the course *Topology*.

2 Homework Exercises

Exercise 1 Let (X, d) be a metric space. Define $B(X) = \{f : X \to \mathbb{R} \mid \exists c \in \mathbb{R} : \forall x \in x |f(x)| < c\}$. Show that $|\cdot|_{C^0(X)} : X \to \mathbb{R}$ defined through $|f|_{C^0(X)} := \sup_{x \in X} |f(x)|$ is a norm on $B(X)$.

Proof The function $|\cdot|_{C^0(X)}$ is well-defined since $|f(X)| = \{|f(x)| \mid x \in X\}$ is a bounded set in $\mathbb{R}$. We show it satisfies the 3 properties of a norm:

1. **homogeneity**: For $\lambda \in \mathbb{R}$, $|\lambda f| = \sup_X |\lambda f(x)| = \sup_X |\lambda||f(x)| = |\lambda| \sup_X |f(x)| = |\lambda||f|$, where in the third equality we use $|\lambda| \geq 0$.
2. **definiteness**: $|f(x)| \geq 0$ for all $x \in X$, so $\sup_X |f(x)| \geq 0$. And, if $|f| = 0$, then $\sup_X |f(x)| = 0$, hence $\forall x \in X : |f(x)| \leq 0$. This implies $\forall x \in X : f(x) = 0$, so $f = 0$, the zero function in $B(X)$, which is also the zero element of $B(X)$.

Exercise 2 Let $\{C_i\}_{i \in I}$ be a family of closed set indexed by an index set I (possibly uncountable). Show that if $J \subset I$ is finite, then

$$\bigcap_{i \in I} C_i \quad \bigcup_{i \in J} C_i$$

are open.

Proof By definition, C_i^c is open for all $i \in I$. Moreover, De Morgan's laws give that

$$\bigcap_{i\in I} C_i = \left[\bigcup_{i\in I} C_i^c\right]^c \quad \bigcup_{i\in J} C_i = \left[\bigcap_{i\in J} C_i^c\right]^c$$

So it is sufficient to show that if $\{O_i\}_{i\in I}$ is a collection of open sets, then for a finite $J \subset I$

$$\bigcap_{i\in J} O_i \quad \bigcup_{i\in I} O_i$$

Are open. This is true by axiom in a topological space $(X, \mathcal{T})$. For a metric space, we define an open set O to be a set $O \subset X$ such that $\forall x \in O : \exists r > 0 : B_r(x) \subset O$.

Then, if $J \subset I$ is finite, and all O_i are open, take a $\bigcap_{i\in J} O_i$. For each i, there is an $r_i > 0$ such that $B_{r_i}(x) \subset O_i$. If we let $r = \min_{i\in J} r_i$, which is well-defined since J is finite, then we see $B_r(x) \subset B_{r_i}(x) \subset O_i$ for all $i \in J$. So there is an open ball around x with radius $r > 0$ that is contained in $\bigcap_{i\in J} O_i$. So $\bigcap_{i\in J} O_i$ is open.

Exercise 3

(i) Let (X, d_X), (Y, d_Y) be metric spaces. Show that $(X \times Y, d)$ where $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$, defined through $d((x, y),(x', y')) := d_X(x, x') + d_Y(y, y')$, is a metric space.

(ii) Show that $X = \prod_{i=1}^{\infty} \mathbb{R}$, endowed with $d : X \times X \to \mathbb{R}$ defined through

$$d(a, b) := \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{2^i[1 + |a_i - b_i|]}$$

is a metric space. First show that d is well-defined (i.e. the series on the right converges).

Proof For (i): d satisfies the axioms of a metric:

1. **definiteness**: $d((x, y),(x', y')) = d_X(x, x') + d_Y(y, y') \geq 0$ where we use that both $d_X(x, x') \geq 0$ and $d_Y(y, y') \geq 0$ because these are metrics. And, if $d(x, y),(x', y')) = 0$, then $d_X(x, x') = d_Y(y, y')$, but since both are nonnegative, this can only happen if $d_X(x, x') = d_Y(y, y') = 0$, and this implies that $x = x'$ and $y = y'$ because d_X and d_Y are definite.

2. **symmetry**: $d((x, y),(x', y')) = d_X(x, x')+d_Y(y, y') = d_X(x', x)+d_Y(y', y) = d((x', y'),(x, y))$, where in the second equality we use symmetry of d_X and d_Y separately, and in the third equality we just substitute the definition of d.

3. **triangle inequality**: For arbitrary $(x'', y'') \in X \times Y$, we have $d((x,y),(x',y')) = d_X(x,x') + d_Y(y,y') \leq d_X(x,x'') + d_Y(y,y'') + d_X(x'',x') + d_Y(y'',y')$, where the inequality follows from the triangle inequality for d_X and d_Y separately. In the final formula, we can substitute the definition of d:... $= d((x,y),(x'',y'')) + d((x',y'),(x'',y''))$. This gives the required inequality.

For (ii): $d(a,b)$ is well-defined, since $\frac{|a_i-b_i|}{1+|a_i-b_i|} < 1$ always, as $t \mapsto \frac{t}{1+t}$ is bounded by $[0,1)$ on $[0,\infty)$. This means that for any $N \in \mathbb{N}$,

$$\sum_{i=1}^{N} \frac{|a_i - b_i|}{2^i[1+|a_i-b_i|]} < \sum_{i=1}^{N} \frac{1}{2^i} = 1 - \frac{1}{2^N}$$

So as $N \to \infty$, the series $d(a,b)$ is bounded by 1, and since it has nonnegative terms, this means that it converges. Next, d satisfies the axioms of a metric:

1. **definiteness**: Each term $\frac{|a_i-b_i|}{2^i[1+|a_i-b_i|]}$ is non-negative for all $i \in \mathbb{N}$, so the series will converge to a nonnegative limit. Moreover, the partial sums are an ascending sequence, so if for any $i \in \mathbb{N}$ it holds $a_i \neq b_i$, then $d(a_i, b_i) > 0$, hence $\frac{|a_i-b_i|}{2^i[1+|a_i-b_i|]} > 0$, then the limit will be strictly positive. So $d(a,b) = 0 \iff \forall i \in \mathbb{N} : a_i = b_i \iff a = b$.

2. **symmetry**: $d(a,b) = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{|a_i-b_i|}{2^i[1+|a_i-b_i|]} = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{d_i(b_i,a_i)}{2^i[1+d_i(b_i,a_i)]} = d(b,a)$ since the equality holds term-wise by symmetry of every d_i.

3. **triangle inequality**: We show that for sequences $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$, $(c_i)_{i\in\mathbb{N}}$ it holds term-wise, that:

$$\frac{|a-b|}{2^i[1+|a-b|]} \leq \frac{|a-c|}{2^i[1+|a-c|]} + \frac{|b-c|}{2^i[1+|b-c|]}$$

Which, dividing by $2^i > 0$ and setting $v = a - c$, $w = b - c$, is equivalent to showing:

$$\frac{|v+w|}{2^i[1+|v+w|]} \leq \frac{|v|}{2^i[1+|v|]} + \frac{|w|}{2^i[1+|w|]}$$

For all $v, w \in \mathbb{R}$. We distinguish two cases:

- If v, w have the same sign, then $|v+w| = |v| + |w|$, so that:

$$\frac{|v+w|}{1+|v+w|} = \frac{|v|}{1+|v|+|w|} + \frac{|w|}{1+|v|+|w|} \leq \frac{|v|}{1+|v|} + \frac{|w|}{1+|w|}$$

- If v, w have opposite sign, then assume w.l.o.g. that $|v| \geq |w|$. Then $|v+w| = |v| - |w| \leq |v|$; moreover, if $0 \leq t \leq s$, then

$$0 \leq \frac{t}{1+t} = 1 - \frac{1}{1+t} \leq 1 - \frac{1}{1+s} = \frac{s}{1+s}$$

Therefore, with $t = |v+w|$ and $s = |v|$:

$$\frac{|v+w|}{1+|v+w|} \leq \frac{|v|}{1+|v|} \leq \frac{|v|}{1+|v|} + \frac{|w|}{1+|w|}$$

This concludes the proof, since we conclude $d(a, b) \leq d(a, c) + d(b, c)$ by the termwise inequality for each $i \in \mathbb{N}$.